SEGNALI

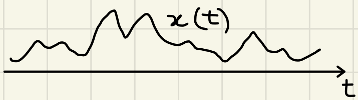
Un segnale è qualunque cosa che porti un'informazione.

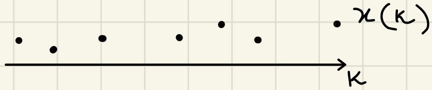
I segnali possono essere:

- <u>monodimensionali</u> VS <u>pluridimensionali</u>

  $V(t)$ $I(t)$ $\vec{E}(x, y, z, t)$

- <u>continui</u> VS <u>discreti</u>





Posso passare da un segnale continuo a uno discreto mediante <u>campionamento</u>

<u>Quantizzazione</u> ($\neq$ discretizzazione):
è il processo con cui si tronca il valore di un'informazione per renderlo finito e memorizzabile (es: $x = \frac{2}{3}$, $x^{(q)} = 0{,}66$)
$\longrightarrow$ introduce un errore trattabile mediante la statistica

<u>deterministico</u> VS <u>casuale</u>

- <u>reale</u> VS <u>complesso</u>

Energia

Si definisce energia di un segnale:

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt$$

Potenza Istantanea

Si definisce potenza istantanea di un segnale: $P_x(t) = |x(t)|^2$

Potenza Media

$$P_x = \frac{\int_{t_1}^{t_2} |x(t)|^2 \, dt}{t_2 - t_1}$$

Segnale periodico

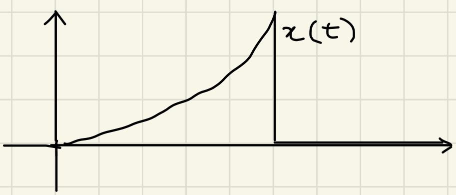
$$x_p(t) = x_p(t - kT)$$
$\forall k \in \mathbb{Z}$ con T periodo

Potenza segnale periodico

$$P_{xp} = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x_p(t)|^2 \, dt$$

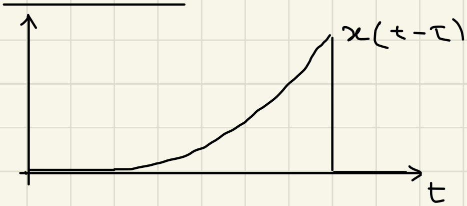
NB: ciascuna espressione é caratterizzata nel caso specifico da un fattore che corregga le unitá di misura

Operazioni fondamentali



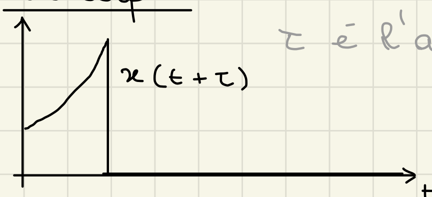
- ritardo

τ é il ritardo

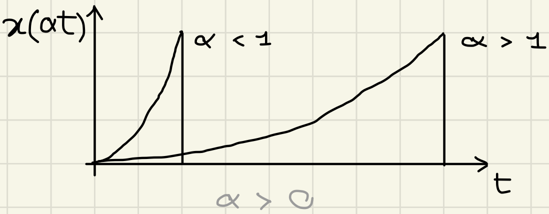


- anticipo

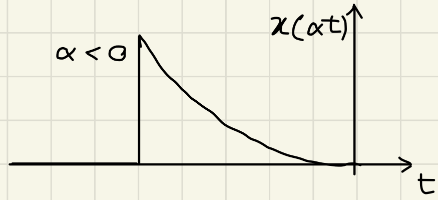
τ é l'anticipo



- scalatura

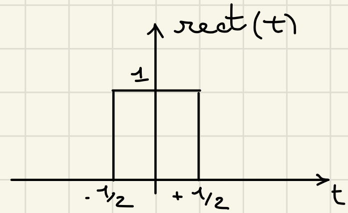


- ribaltamento



Segnali rettangolari

$$\mathrm{rect}(t) = \begin{cases} 1 & |t| < 1/2 \\ 0 & |t| > 1/2 \end{cases}$$



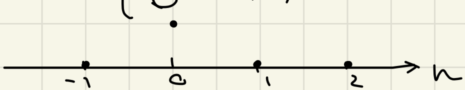
- somma
- sottrazione
- prodotto
- divisione

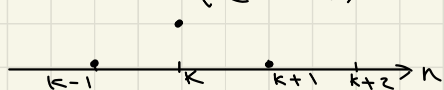
Rappresentazione segnale complesso:
- parte reale e parte immaginaria
- modulo e fase (polare)

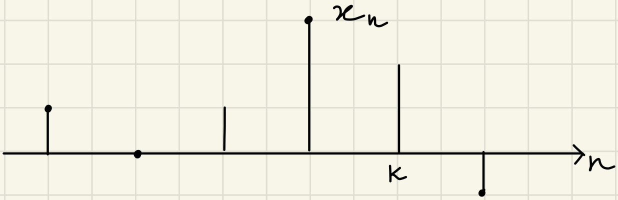
discreto ↑ **Delta di Kronecker** δ_n

continuo ↑ **Delta di Dirac** $\delta(t)$

$$\delta_n = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \qquad \sum_n \delta_n = 1 \qquad \delta_{n-k} = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases}$$

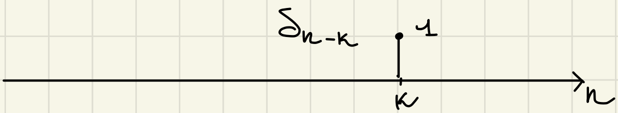






$$\boxed{x_n \delta_{n-k} = x_k \delta_{n-k}}$$

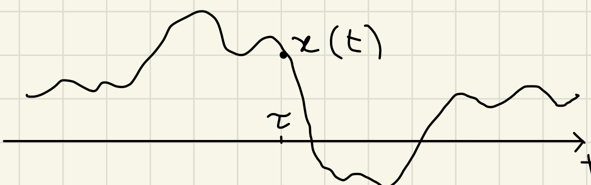
$$\sum_k x_n \delta_{n-k} = \sum_k x_k \delta_{n-k}$$



$$x_n \underbrace{\sum_k \delta_{n-k}}_{1} = \sum_k x_k \delta_{n-k}$$

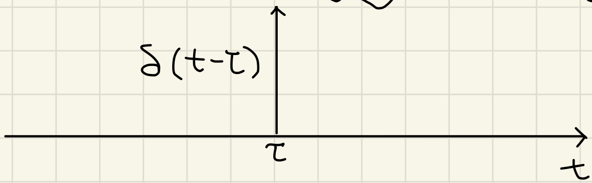
$$\Rightarrow \boxed{x_n = \sum_k x_k \delta_{n-k}}$$

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases} \qquad \int \delta(t)\,dt = 1 \qquad \delta(t-\tau) = \begin{cases} \infty & t = \tau \\ 0 & t \neq \tau \end{cases}$$

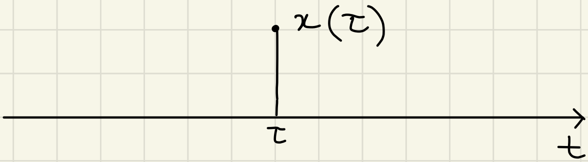


$$\boxed{x(t)\delta(t-\tau) = x(\tau)\delta(t-\tau)}$$

$$\int x(t)\delta(t-\tau)\,d\tau = \int x(\tau)\delta(t-\tau)\,d\tau$$

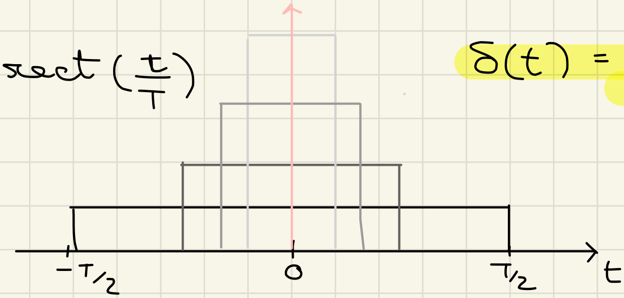


$$x(t) \overbrace{\int \delta(t-\tau)\,d\tau}^{1} = \int x(\tau)\delta(t-\tau)\,d\tau$$

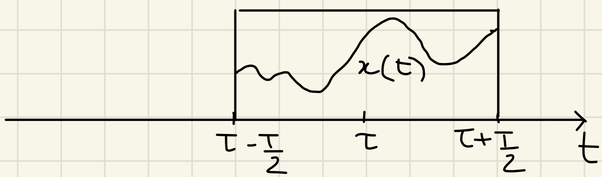


$$\Rightarrow \boxed{x(t) = \int x(\tau)\delta(t-\tau)\,d\tau}$$

$$\frac{1}{T}\,\mathrm{rect}\left(\frac{t}{T}\right) \qquad\qquad \delta(t) = \lim_{T\to 0} \frac{1}{T}\,\mathrm{rect}\left(\frac{t}{T}\right)$$



$$\frac{1}{T}\,\mathrm{rect}\left(\frac{t-\tau}{T}\right) \qquad\qquad \int x(t)\,\frac{1}{T}\,\mathrm{rect}\left(\frac{t-\tau}{T}\right)d\tau = \;= x(\tau)$$



media integrale di $x(t)$

$T \to 0$

$t \to \tau$

$\tau \to t$

$$\left[\int x(\tau)\,\delta(t-\tau)\,dt = x(t)\right]$$

SISTEMI LINEARI TEMPO INVARIANTI (LTI)

tempo continuo: $\quad x(t) \to \boxed{\text{LTI}} \to y(t)$

- Linearità

$y(t) = L(x(t))$

$\Downarrow$

sovrapposizione degli effetti

L può essere $\int x(t)\,dt$, $\frac{d^n x(t)}{dt^n}$

non può essere $x^2(t)$

$$\sum_n a_n x_n(t) \to \boxed{L} \to \sum_n a_n y_n(t)$$

- Tempo invarianza

$$x(t-\tau) \to \boxed{TI} \to y(t-\tau)$$

Se conosco $h(t)$ = risposta all'impulso ovvero

$$\delta(t) \to \boxed{LTI} \to h(t)$$

posso calcolare la risposta a un qualunque segnale $x(t) \to \boxed{LTI} \to y(t)$?

$$\delta(t-\tau) \to \boxed{} \to h(t-\tau)$$

$$x(\tau)\delta(t-\tau)d\tau \to \boxed{} \to x(\tau)h(t-\tau)d\tau$$

$$x(t) = \int x(\tau)\delta(t-\tau)d\tau \to \boxed{} \to \int x(\tau)h(t-\tau)d\tau$$

$$x(t) \to \boxed{} \to y(t)$$

$$\Longrightarrow \left[y(t) = \int x(\tau)h(t-\tau)d\tau \right] \quad \forall x(t)$$

CONVOLUZIONE a t. continuo

$$\left[y(t) = x(t) * h(t) \right]$$

tempo discreto:

$$x_n \to \boxed{LTI} \to y_n$$

h_n risposta all'impulso $\delta_n \to \boxed{LTI} \to h_n$

$$\delta_{n-k} \to \boxed{} \to h_{n-k}$$

$$x_k \delta_{n-k} \to \boxed{} \to x_k h_{n-k}$$

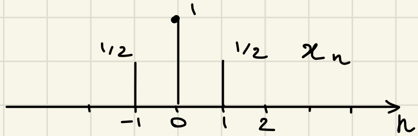
$$x_n = \sum_k x_k \delta_{n-k} \to \boxed{} \to \sum_k x_k h_{n-k}$$

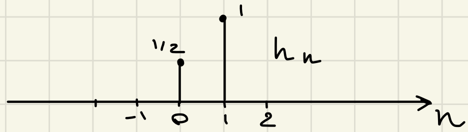
$$\Longrightarrow \left[y_n = \sum_k x_k h_{n-k} \right]$$

CONVOLUZIONE a t. discreto

$$\left[y_n = x_n * h_n \right]$$

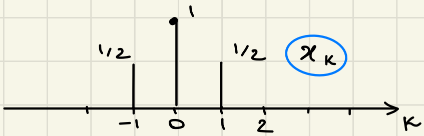
Es.:



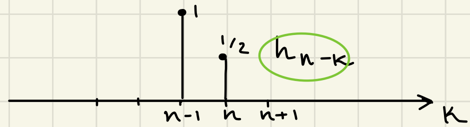


$$y_n = ?$$

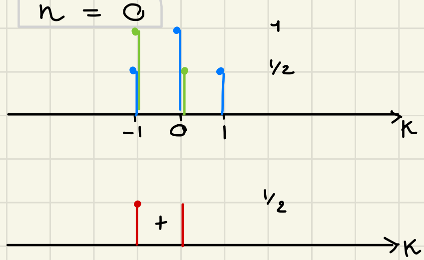
$$y_n = x_n * h_n = \sum_k x_k h_{n-k}$$



$*$



$\Longrightarrow$ $\boxed{n = 0}$

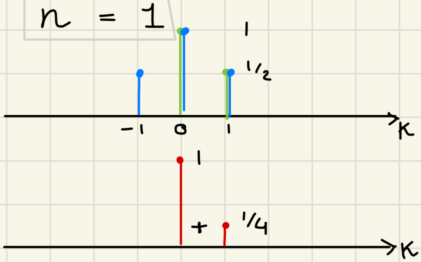


$$y_0 = \sum_k x_k h_{-k}$$

$$= \frac{1}{2} \cdot 1 + 1 \frac{1}{2} + \frac{1}{2} \cdot 0$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

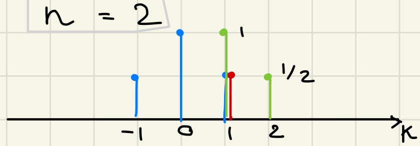
$\Longrightarrow$ $\boxed{n = 1}$



$$y_1 = 1 \cdot 1 + \frac{1}{2} \cdot \frac{1}{2}$$

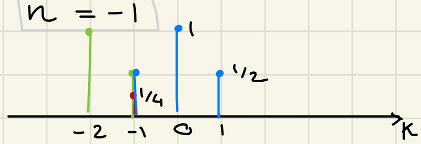
$$= 1,25$$

$\Longrightarrow$ $\boxed{n = 2}$



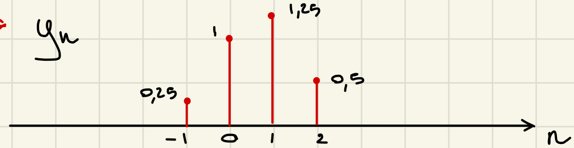
$$y_2 = 0,5$$

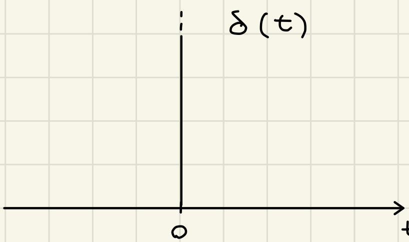
$\Longrightarrow$ $\boxed{n = -1}$

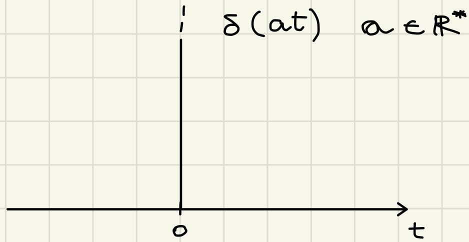


$$y_{-1} = 0,25$$

$\Rightarrow$



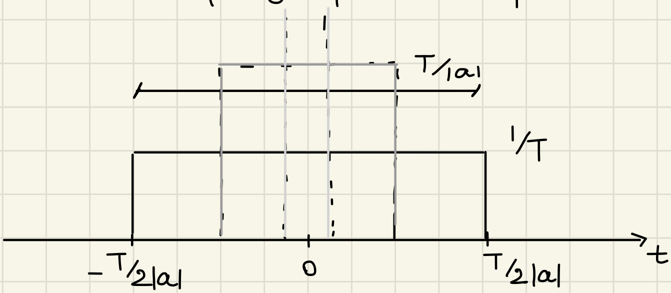




$\int \delta(t)\,dt = 1$

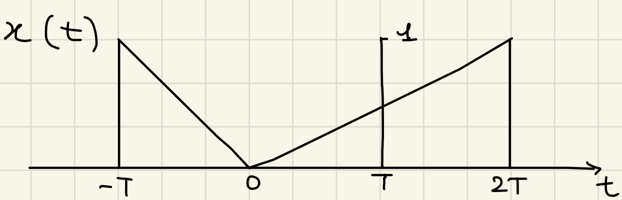
$\int \delta(at)\,dt = ?$

$$\delta(at) = \lim_{T \to 0} \frac{1}{T}\, \mathrm{rect}\left(\frac{at}{T}\right)$$

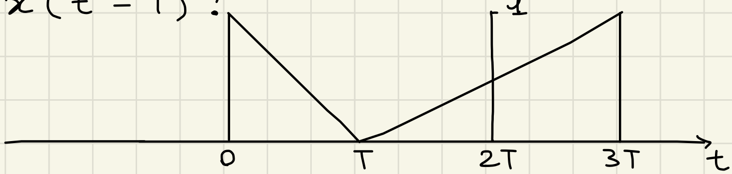


$\Rightarrow \int \delta(at)\,dt = 1/|a|$

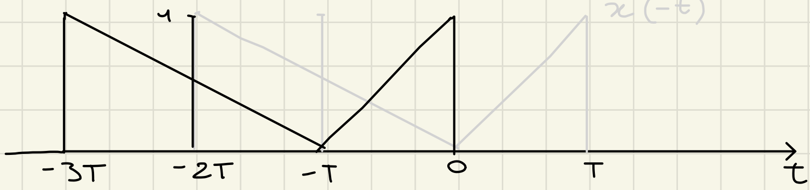
Esercizio.



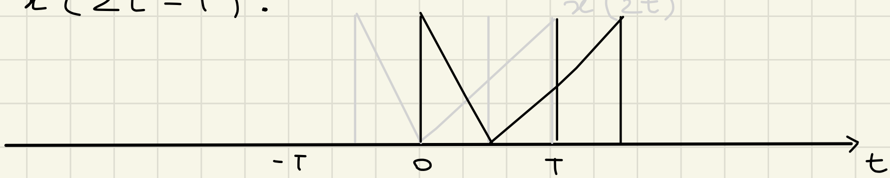
$x(t-T)$?



$x(-t-T)$?



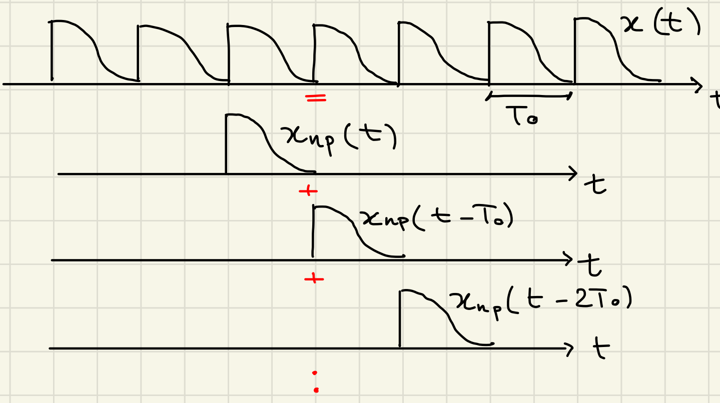
$x(2t-T)$?



Potenza segnale periodico

$x(t) = x(t + kT_0) \quad \forall k \in \mathbb{Z}$ segnale periodico

$x(t) = \sum_n x_{np}(t - nT_0)$



$$P = \lim_{T_{oss}\to\infty} \frac{1}{T_{oss}} \int_{-T_{oss}/2}^{T_{oss}/2} |x(t)|^2 \, dt$$

$$(x_1+x_2)^2 = x_1^2 + 2x_1x_2 + x_2^2 = \sum_{i,j} x_i x_j \qquad x \in \mathbb{R}$$

$$\boxed{P = \frac{1}{T_0} \int_{T_0} |x(t)|^2 \, dt \quad \text{segnali periodici}}$$

$$|x_1+x_2|^2 = |x_1|^2 + |x_2|^2 + 2\,\mathrm{Re}(x_1\bar{x}_2) = \sum_{i,j} x_i \bar{x}_j \qquad x \in \mathbb{C}$$

$$|x(t)|^2 = \left|\sum_n x_{np}(t-nT_0)\right|^2 = \sum_n |x_{np}(t-nT_0)|^2$$

$$= \sum_{n,m} x_{np}(t-nT_0)\,\bar{x}_{mp}(t-mT_0)$$

$$P = \frac{1}{T_{oss}} \int \sum_n |x_{np}(t-nT_0)|^2 = \frac{1}{T_{oss}} (N E_{np} + \alpha E_{np})$$

$$T_{oss} = N T_0 + \alpha T_0 \qquad \alpha \in (0,1)$$

$$= \frac{N}{T_{oss}} E_{np} + \frac{\alpha}{T_{oss}} E_{np}$$

$$\Big\downarrow \; T_{oss} \to \infty$$

$$P = \frac{1}{T_0} E_{np} + 0$$

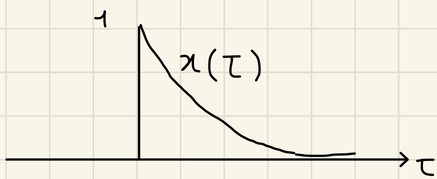
$$= \frac{1}{T_0} \int_{T_0} |x(t)|^2 \, dt$$

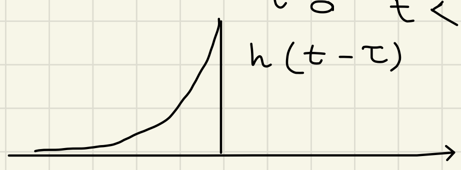
Esercizi.

- $y(t) = \int [x(\tau) \cdot h(t-\tau)] \, d\tau$

$$x(t) = h(t) = e^{-at} u(t) \qquad a > 1$$

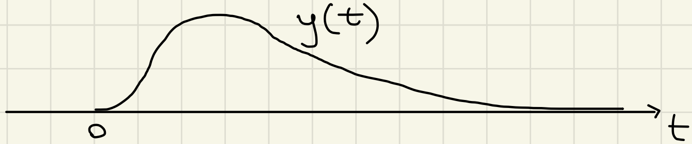
$$u(t) = \begin{cases} 1 & t > 0 \\ 1/2 & t = 0 \\ 0 & t < 0 \end{cases}$$





$$y(t) = \int e^{-a\tau} u(\tau)\, e^{-a(t-\tau)} u(t-\tau)\, d\tau =$$

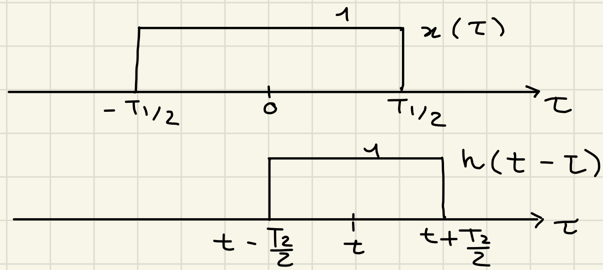
$$= \int_0^t e^{-at}\, d\tau = t\, e^{-at} \qquad t > 0$$



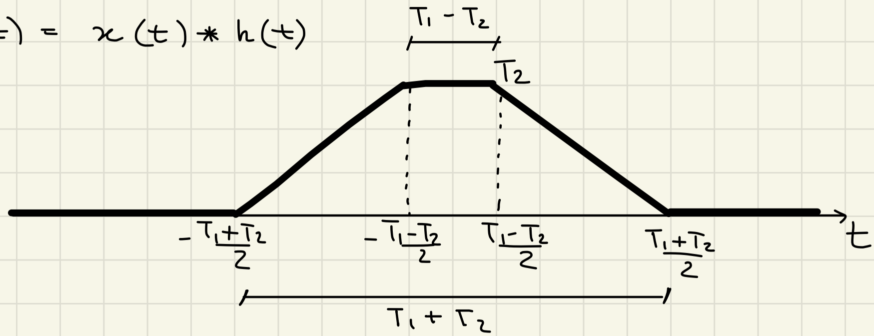
- $x(t) = \mathrm{rect}\left(\frac{t}{T_1}\right)$

  $h(t) = \mathrm{rect}\left(\frac{t}{T_2}\right)$

  $T_1 > T_2$



$y(t) = x(t) * h(t)$

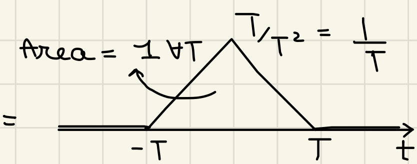


$\boxed{T_y \leq T_x + T_h}$ $\boxed{\int y(t)\,dt = \int x(t)\,dt \cdot \int h(t)\,dt}$

- $\delta(t) * \delta(t) = ?$ $\qquad (\delta_n * \delta_n = \delta_n)$

  $\delta(t) = \lim_{T \to 0} \frac{1}{T}\, \mathrm{rect}\left(\frac{t}{T}\right)$

  $\frac{1}{T}\, \mathrm{rect}\left(\frac{t}{T}\right) * \frac{1}{T}\, \mathrm{rect}\left(\frac{1}{T}\right) =$ per $T \to 0$



$$\Longrightarrow \boxed{\delta(t) * \delta(t) = \delta(t)}$$

Proprietà della convoluzione

- Associativa

$$x(t) * (y(t) * z(t)) = (x(t) * y(t)) * z(t)$$

- Distributiva

$$x(t) * (y(t) + z(t)) = x(t) * y(t) + x(t) * z(t)$$

- Commutativa

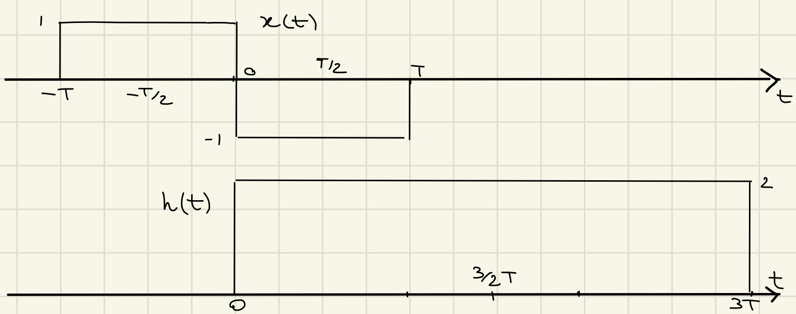
$$x(t) * h(t) = h(t) * x(t)$$

Esercizi:

- $$x(t) = \text{rect}\left(\frac{t + T/2}{T}\right) - \text{rect}\left(\frac{t - T/2}{T}\right)$$

$$h(t) = 2\,\text{rect}\left(\frac{t - 3/2\,T}{3T}\right)$$

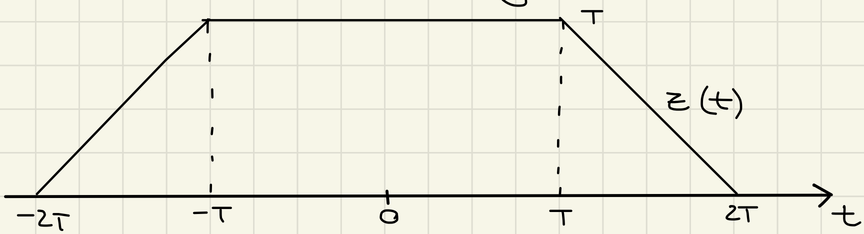
$$y(t) = x(t) * h(t)$$



Pongo $s(t) = \mathrm{rect}(t/T)$
$g(t) = \mathrm{rect}(t/3T)$

$$\Longmapsto \quad y(t) = 2\left[\left(s\left(t+\frac{T}{2}\right) - s\left(t-\frac{T}{2}\right)\right) * g\left(t-\frac{3}{2}T\right)\right]$$

$$z(t) = s(t) * g(t)$$

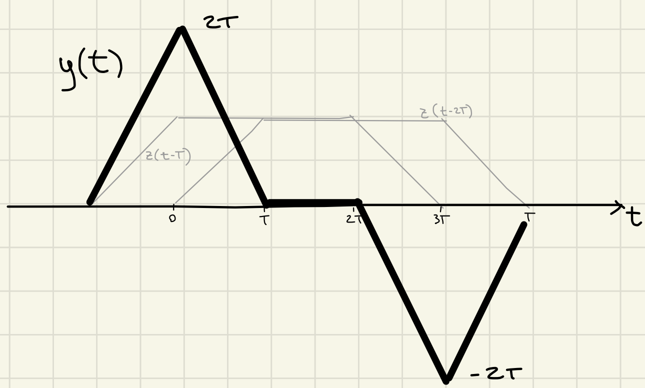


$$s(t) \longrightarrow \boxed{g(t)} \longrightarrow z(t)$$
$$s\left(t+\frac{T}{2}\right) \longrightarrow \boxed{g(t)} \longrightarrow z\left(t+\frac{T}{2}\right)$$
$$g(t) \longrightarrow \boxed{s\left(t+\frac{T}{2}\right)} \longrightarrow z\left(t+\frac{T}{2}\right)$$
$$g\left(t-\frac{3}{2}T\right) \longrightarrow \boxed{s\left(t+\frac{T}{2}\right)} \longrightarrow z\left(t+\frac{T}{2}-\frac{3}{2}T\right)$$
$$\Rightarrow \quad g\left(t-\frac{3}{2}T\right) * s\left(t+\frac{T}{2}\right) = z(t-T)$$

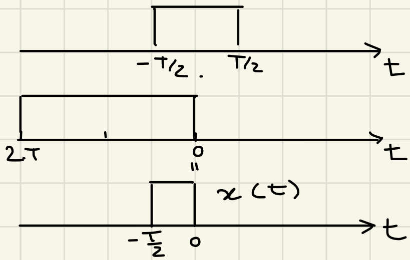
. . . . .

$$g\left(t-\frac{3}{2}T\right) \longrightarrow \boxed{s\left(t-\frac{T}{2}\right)} \longrightarrow z\left(t-\frac{T}{2}-\frac{3}{2}T\right)$$
$$\Rightarrow \quad g\left(t-\frac{3}{2}T\right) * s\left(t+\frac{T}{2}\right) = z(t-2T)$$

$$\Longrightarrow \quad y(t) = 2\left[z(t-T) - z(t-2T)\right]$$

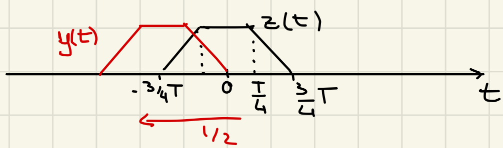


- $$y(t) = \Big(\overbrace{\mathrm{rect}\left(\tfrac{t}{T}\right)\mathrm{rect}\Big(\underbrace{\tfrac{t}{2T}+\tfrac{1}{2}}_{\frac{t+T}{2T}}\Big)}^{x(t)}\Big) * \mathrm{rect}\left(\tfrac{t}{T}\right)$$



$$x(t) = \mathrm{rect}\left(\frac{t+T/4}{T/2}\right)$$

$$z(t) = \mathrm{rect}\left(\frac{t}{T/2}\right) * \mathrm{rect}\left(\frac{t}{T}\right)$$



$$\mathrm{rect}\left(\frac{t}{T/2}\right) \longrightarrow \boxed{\mathrm{rect}\left(\tfrac{t}{T}\right)} \longrightarrow z(t)$$

$$\mathrm{rect}\left(\frac{t}{T/2}+\frac{1}{2}\right) \longrightarrow \boxed{\mathrm{rect}\left(\tfrac{t}{T}\right)} \longrightarrow z\left(t+\tfrac{1}{2}\right) = y(t)$$

- $$y(t) = \underbrace{x(t) * \left[a\,\delta(t-\tau_1)\right.}\left. + b\,\delta(t-\tau_2)\right]$$

$$\int x(t-\tau)\cdot a\,\delta(\tau-\tau_1)\,d\tau = a\int x(t-\tau)\,\delta(\tau-\tau_1)\,d\tau$$
$$= a\,x(t-\tau_1)\int\delta(\tau-\tau_1)\,d\tau = a\,x(t-\tau_1)$$

$$\Rightarrow\ y(t) = a\,x(t-\tau_1) + b\,x(t-\tau_2)$$

$$e^{j2\pi f t} \longrightarrow \boxed{h(t)} \longrightarrow \int e^{j2\pi f\tau}\cdot h(t-\tau)\,d\tau =$$
$$= \int e^{j2\pi f(t-\tau)}\,h(\tau)\,d\tau =$$
$$= e^{j2\pi f t}\underbrace{\int h(\tau)\,e^{-j2\pi f\tau}\,d\tau}_{H(f)\ \text{risposta in frequenza}}$$

$$\cos(2\pi f t) = \frac{e^{j2\pi f t} + e^{-j2\pi f t}}{2} \longrightarrow \boxed{h(t)} \longrightarrow \frac{1}{2} e^{j2\pi f t} H(f) + \frac{1}{2} e^{-j2\pi f} H(-f)$$

$$H^*(f) = \int h(\tau)\, e^{+j2\pi f \tau}\, d\tau \quad \text{se } h(t) \in \mathbb{R}$$

$$H(-f) = \int h(\tau)\, e^{+j2\pi f \tau}\, d\tau$$

$$h(t) \in \mathbb{R} \implies H^*(f) = H(-f)$$

$$H(f) = |H(f)|\, e^{j \measuredangle H(f)}$$

Trasformate di Fourier

$$e^{j2\pi f t} \longrightarrow \boxed{h(t)} \longrightarrow H(f)\, e^{j2\pi f t}$$

$$\parallel$$

$$|H(f)|\, e^{j2\pi f (t + \measuredangle H(f))}$$

$$H(f) = \int h(t)\, e^{-j2\pi f t}\, dt$$

$$\left.\begin{aligned} s(t) &= A\, e^{j(2\pi f t + \vartheta)} \\ s(t) &= A \cos(2\pi f t + \vartheta) \end{aligned}\right\} \text{segnale generico}$$

$$F(x(t)) = \int x(t)\, e^{-j2\pi f t}\, dt = X(f)$$

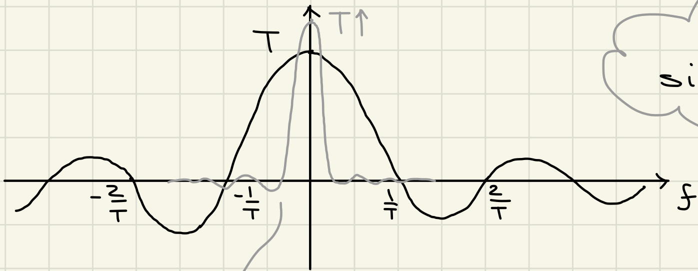
- $F(\delta(t)) = \int \delta(t)\, \underbrace{e^{-j2\pi f t}}_{=1 \text{ se } t=0}\, dt = \int \delta(t)\, dt = \underline{1}$

$$x(t)\,\delta(t) = x(0)\,\delta(t)$$

- $\mathcal{F}(\text{rect}(\frac{t}{T})) = X(f)$

$$X(f) = \int_{-T/2}^{T/2} e^{-j2\pi f t}\, dt = \frac{\left[e^{-j2\pi f t}\right]_{-T/2}^{T/2}}{-j2\pi f} = \frac{e^{j\pi T f} - e^{-j\pi T f}}{j2\pi f}$$

$$\boxed{X(f) = \frac{\sin(\pi f T)}{\pi f} = T\,\text{sinc}(fT)}$$



seno cardinale

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

per T↑ il grafico di $X(f)$ tende a quello di $\delta(f)$

$$A = \int \frac{\sin(\pi f T)}{\pi f}\, df = \int \frac{\sin(\pi \upsilon)}{\pi \upsilon} \cancel{\frac{T}{T}}\, d\upsilon$$

$fT = \upsilon$

$\Downarrow$

1

1

$$\frac{\sin(\pi f T)}{\pi f} \xrightarrow[T\to\infty]{} \cancel{A}\,\delta(f)$$

$\text{rect}(\frac{t}{\infty})$

- $\boxed{\mathcal{F}(1) = \int e^{-j2\pi f t}\, dt = \delta(f)}$

Trasformata inversa

$$\int X(f)\, e^{+j2\pi f t}\, df = \iint x(\tau)\, e^{-j2\pi f(\tau - t)}\, d\tau\, df$$

$$= \int x(\tau)\left(\int e^{-j2\pi f(\tau-t)}\, df\right) d\tau$$

$$X(f) = \int x(\tau)\, e^{-j2\pi f \tau}\, d\tau$$

$$\int e^{-j2\pi f(\tau-t)} = \delta(\tau - t) = \delta(t-\tau)$$

$$\Longrightarrow \int x(\tau) \int e^{-j2\pi f(\tau-t)} \, df \, d\tau$$

$$= \int x(\tau)\, \delta(t-\tau)\, d\tau = x(t)$$

$$\boxed{\begin{aligned} x(t) &= \int X(f)\, e^{+j2\pi f t}\, df \\ X(f) &= \int x(t)\, e^{-j2\pi f t}\, dt \end{aligned}}$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

$$\int X(f)\, e^{+j2\pi f t}\, df \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \int H(f)\, X(f)\, e^{j2\pi f t}\, df$$

$$\left[y(t) = \int Y(f)\, e^{+j2\pi f t}\, df, \quad Y(f) = H(f)\, X(f) \right]$$

Proprietà

1) **Linearità**

$$F(a x(t) + b y(t)) = a X(f) + b Y(f)$$

2) **Ritardo**

$$F(x(t-\tau)) = X(f)\, e^{-j2\pi f \tau}$$

3) **Ritardo nelle frequenze**

$$F(x(t)\, e^{+j2\pi f_0 t}) = X(f - f_0)$$

4) **Dualità**

$$x(t) \Longleftrightarrow X(f), \qquad X(-t) \Longleftrightarrow x(f)$$

es: $F\left(\frac{\sin(\pi t B)}{\pi t}\right) = rect\left(\frac{f}{B}\right)$ Banda

5) Modulazione

$F(x(t)\cdot y(t)) = X(f) * Y(f)$

es: $F(x(t)e^{+j2\pi f_0 t}) = X(f-f_0) = X(f) * \delta(f-f_0)$

infatti $F(e^{j2\pi f_0 t}) = \int e^{-j2\pi(f-f_0)\sigma} d\sigma = \delta(f-f_0)$

6) Scalatura

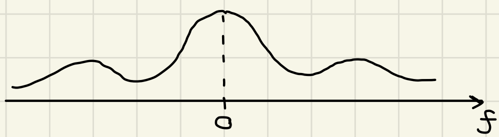
$F(x(at)) = \frac{1}{|a|} X\left(\frac{f}{a}\right) \longrightarrow F(x(-t)) = X(-f)$

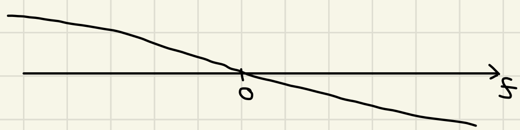
7) Complesso coniugato

$F(x^*(t)) = X^*(-f)$

↳ Se $x(t) \in \mathbb{R} \Rightarrow x^*(t) = x(t) \Rightarrow X(-f) = X^*(f)$

$|X(f)| = |X(-f)|$ $\qquad$ $\angle X(f) = -\angle X(-f)$





modulo PARI $\qquad$ fase DISPARI

8) Derivata

$F\left(\frac{d}{dt} x(t)\right) = j2\pi f X(f)$

(correlazione tra il valore di 2 segnali a distanza temporale τ)

Cross - correlazione

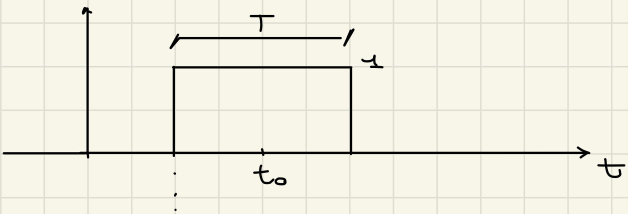
$$\left[R_{xy}(\tau) = \int x(t+\tau)\cdot y^*(t)\,dt = x(\tau) * y^*(-\tau) \right]$$

<u>Dim</u>: $x(\tau) * y^*(-\tau) = \int x(t)\, y^*(-(\tau-t))\,dt =$ $\quad t = \upsilon+\tau$

$$= \int x(\upsilon+\tau)\, y^*(-(\tau-\upsilon-\tau))\,d\upsilon = \int x(t+\tau)\, y^*(t)\,dt$$

$$\left[F(R_{yx}(t)) = Y(f)\cdot X^*(f) = S_{yx}(f) \right]$$

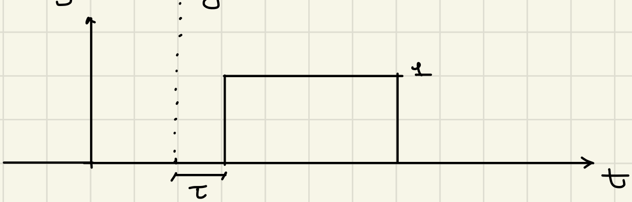
$x(t)$ segnale trasmesso



$$X(f) = \frac{\sin(\pi f T)}{\pi f}\, e^{-j2\pi f t_0}$$

$$|X(f)|^2 = \left(\frac{\sin(\pi f t)}{\pi f} \right)^2$$

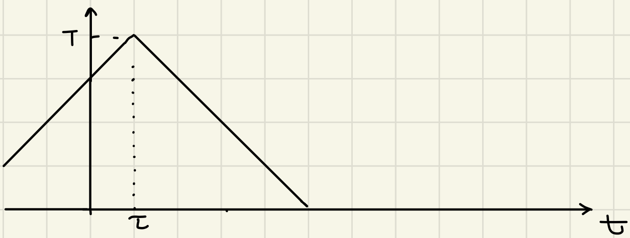
$y(t)$ segnale ricevuto



$$y = x(t-\tau)$$

$$Y(f) = X(f)\, e^{-j2\pi f \tau}$$

$R_{xy}(t)$



$$R_{yx}(t) = T\,\mathrm{tri}\left(\frac{t-\tau}{2T}\right)$$

$$\begin{aligned} S_{xy}(f) &= Y(f)\cdot X^*(f) \\ &= X(f)\, e^{-j2\pi f \tau}\cdot X^*(f) \\ &= |X(f)|^2 e^{-j2\pi f \tau} \end{aligned}$$

massima correlazione alla distanza temporale τ

$$\left\{ R_{xy}(\tau) = R^*_{xy}(\tau) \right.$$

Auto - correlazione

(correlazione tra 2 valori dello stesso segnale a distanza temporale τ)

$$\boxed{R_{xx}(\tau) = x(\tau) * x^*(-\tau) = \int x(t+\tau) \cdot x^*(t)\,dt}$$

$$\boxed{S_{xx}(f) = |X(f)|^2} \qquad \boxed{R_{xx}(0) = \int |x(t)|^2 dt = E_x}$$

$$|R_{xx}(t)| \leq R_{xx}(0)$$

$\left| \int dt \right| \leq \int | \; | dt$

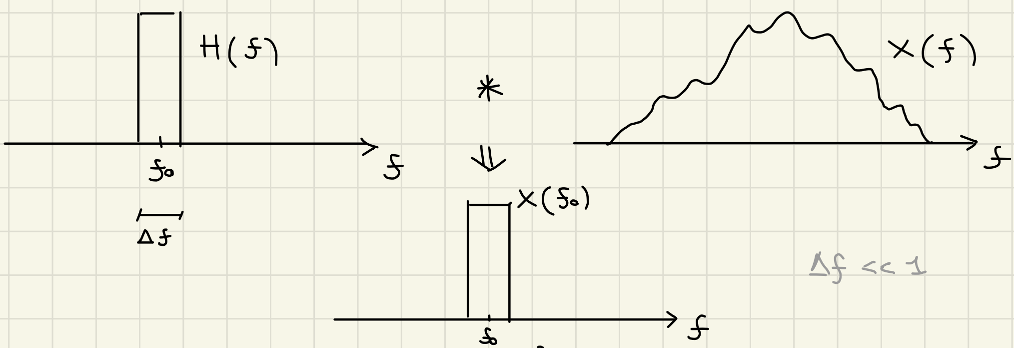
$\longrightarrow$ Energia

$S_{xx}(f) = |X(f)|^2 =$ densità spettrale di energia $\left[\frac{J}{Hz}\right]$

$\left(x = [\sqrt{W}] \right)$

$$E_x(f_0, \Delta f) = E_y$$

$x(t) * h(t) = y(t)$ $\qquad\qquad$ $Y(f) = X(f) \cdot H(f)$



$\Delta f << 1$

$$E_y = \int |Y(f)|^2 df = \int_{f_0 - \frac{\Delta f}{2}}^{f_0 + \frac{\Delta f}{2}} |X(f)|^2 df = \Delta f \, |X(f_0)|^2$$

$\int |x(t)|^2 dt = \int |X(f)|^2 df \rightarrow$ caso particolare del teo. di Parseval

$(y(t) = x(t))$

Teo. di Parseval: $\int y(t)\, x^*(t)\, dt = \int Y(f)\, X^*(f)\, df$ oppure $\int y(t)\, x(-t)\, dt = \int Y(f)\, X(f)\, df$

Es. • $x(t) = \dfrac{\sin(\pi t B)}{\pi t}$ $\qquad X(f) = \mathrm{rect}\left(\dfrac{f}{B}\right)$

$$E_x = \int \left(\frac{\sin(\pi t B)}{\pi t}\right)^2 dt = \int \mathrm{rect}^2\left(\frac{f}{B}\right) df$$

$$= \int_{-B/2}^{B/2} df = B$$

• $x(t) = \left(\dfrac{\sin(\pi t B)}{\pi t}\right)^2$ $\qquad X(f) = \mathrm{rect}\left(\dfrac{f}{B}\right) * \mathrm{rect}\left(\dfrac{f}{B}\right) = B\, \mathrm{tri}\left(\dfrac{f}{B}\right)$

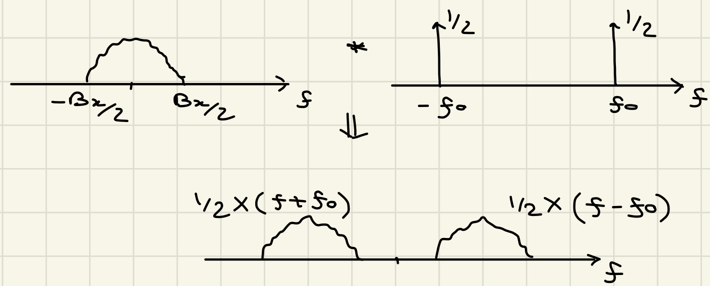
$$E_x = \int \left(\frac{\sin(\pi t B)}{\pi t}\right)^4 dt =$$

$$= \int |X(f)|^2 df = 2\int_0^{B/2} (B-2f)^2 df = 2\left[fB^2 + \frac{4}{3}f^3 - f^2B\right]_0^{B/2}$$

$$= B^3 + \frac{4}{12}B^3 - \frac{B^3}{4} = \frac{13}{12}B^3$$

• $s(t) = x(t)\cos(2\pi f_0 t)$ $\qquad f_0 > \dfrac{B_x}{2}$

$$E_s = \int x^2(t)\cos^2(2\pi f_0 t)\, dt = ?$$



$$S(f) = \frac{1}{2}\left(X(f-f_0) + X(f+f_0)\right)$$

$$|S(f)|^2 = \frac{1}{4}\left(|X(f-f_0)|^2 + |X(f+f_0)|^2 + \underbrace{2\,\mathrm{Re}\left[X(f-f_0)X^*(f+f_0)\right]}_{0}\right)$$

x e y in BANDA BASE (centrati in $f=0$)

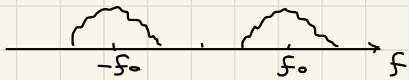
• $s(t) = x(t)\cos(2\pi f_0 t) - y(t)\sin(2\pi f_0 t)$

$$E_s = \int \Big[\overbrace{x^2(t)\cos^2(2\pi f_0 t)}^{E_x/2} + \overbrace{y^2(t)\sin^2(2\pi f_0 t)}^{E_y/2} - 2\,x(t)\,y(t)\cos(2\pi f_0 t)\sin(2\pi f_0 t)\Big]dt$$

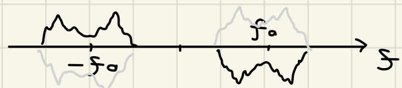
$$\int x(t)\cos(2\pi f_0 t)\, y(t)\sin(2\pi f_0 t)\,dt = \int F\big(x(t)\cos(2\pi f_0 t)\big)\, F^*\big(y(t)\sin(2\pi f_0 t)\big)\,df$$

teo. di Parseval

$\rightarrow F(x\cdot\cos) =$



$\rightarrow F^*(y\cdot\sin) =$



$\Rightarrow \int F(x\cdot\cos)\,F^*(y\sin)\,df = 0$ perché le 2 aree in corrispondenza di $-f_0$ e f_0 sono uguali e opposte.

$\longrightarrow$ Potenza

$x(t) = A\,e^{j2\pi f_0 t}$

$$P_x = \lim_{T\to+\infty} \frac{1}{T}\int_T |x(t)|^2 dt \overset{\text{se periodico}}{=} \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt$$

$P_x = |A|^2$

$x(t) = A_1 e^{j2\pi f_1 t} + A_2 e^{j2\pi f_2 t} \qquad f_1 \neq f_2$

$|x(t)|^2 = |A_1|^2 + |A_2|^2 + 2\,\mathrm{Re}\left[A_1 A_2^* \cdot e^{j2\pi(f_1-f_2)t}\right]$

$P_x = |A_1|^2 + |A_2|^2 + 2\,\mathrm{Re}\left[\lim_{T\to\infty}\frac{1}{T} A_1 A_2^* \int e^{j2\pi(f_1-f_2)t}dt\right]$

$= |A_1|^2 + |A_2|^2$

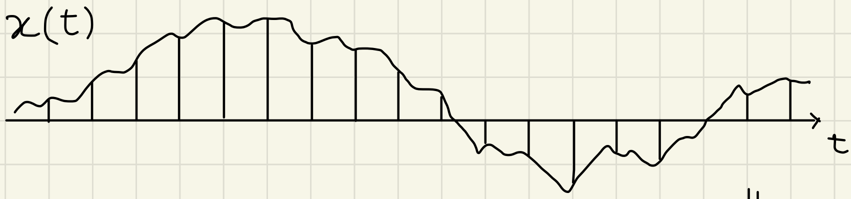
$F(\delta) = 1 \leftrightarrow F(1) = \delta(f) = \int e^{-j2\pi f \tau} d\tau$

$\int \delta(f_1 - f_2) = 1$

$$\Longrightarrow \quad x(t) = \sum_k A_k e^{j2\pi f_k t} \rightarrow X(f) = \sum_k A_k \delta(f - f_k)$$

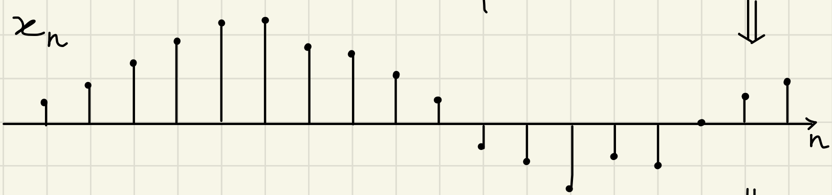
$$\left[P_x = \sum_k A_k^2 \right]$$ ortogonalità in potenza

Teorema del Campionamento

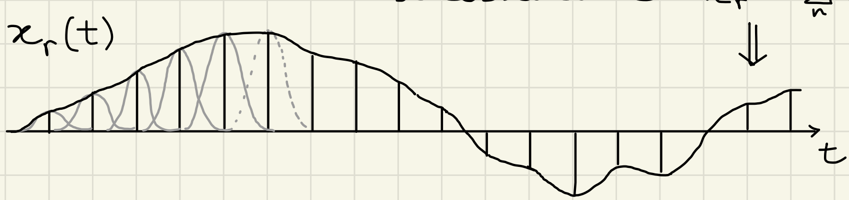


campionamento $x_n = x(t = nT)$

tempo di campionamento



ricostruzione $x_r = \sum_n x_n h(t - nT)$



A cosa serve?

* Trasmissione
* Archiviazione
* Calcolo numerico
* Segnali composti e/o periodici nel tempo e nelle frequenze

$x(t)$

$X(f) = \int x(t)\, e^{-j2\pi f t}\, dt$

$x_n = x(t=nT)$

$X_c(f) = \sum_n x_n e^{-j2\pi f nT}$

?

$$x_c(t) = x(t)\cdot \sum_n \delta(t-nT) = \sum_n \overbrace{x(t=nT)}^{x_n}\,\delta(t-nT)$$

↳ equivalente continuo del segnale campionato discreto

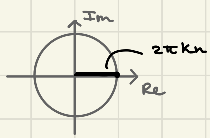
$$X_c(f) = \int \underbrace{x_c(t)}_{=0\ \forall t \neq nT} e^{-j2\pi f t}\, dt = \sum_n x_n e^{-j2\pi f nT}$$

$$X_c(f) = \sum_n x_n e^{-j2\pi f nT} = X(f) * F\Big(\sum_n \delta(t-nT)\Big)$$

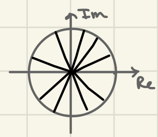
$$F\Big(\sum_n \delta(t-nT)\Big) = \frac{1}{T}\sum_k \delta\Big(f-\frac{k}{T}\Big)$$

↳ Dimostrazione:

- $f = \frac{k}{T}$ $\quad k \in \mathbb{Z}$ $\Rightarrow$ $\sum_n e^{-j2\pi kn} = \infty$

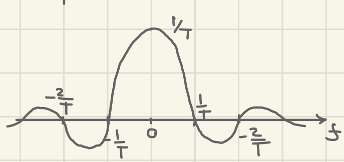


$f \neq \frac{k}{T}$ $\Rightarrow$ $\sum_n = 0$



$$A = \int_{-1/2T}^{1/2T} \sum_n e^{-j2\pi f nT}\, df$$

$$A = \sum_n \frac{\sin(\pi n)}{\pi n T} \rightarrow$$

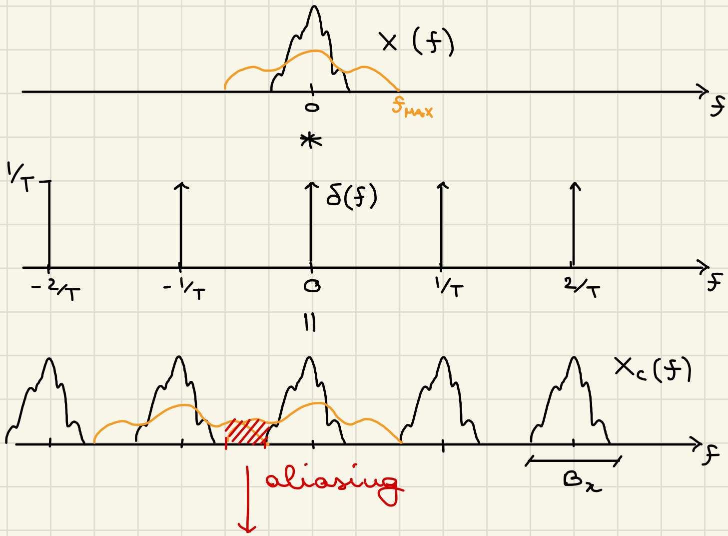


$$\boxed{F\Big(\sum_n \delta(t-nT)\Big) = \frac{1}{T}\sum_k \delta\Big(f-\frac{k}{T}\Big)}$$

- $$\sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} e^{-j2\pi f n T} = e^{+j\pi f(N-1)T} \underbrace{\sum_{n=0}^{N-1} e^{-j2\pi f n T}}_{S_N \text{ serie geometrica}} \qquad S_n = \sum_{n}^{N} e^{-\alpha n} = \frac{1-e^{-\alpha N}}{1-e^{-\alpha}}$$

$$= e^{j\pi f(N-1)T} \frac{1-e^{-j2\pi f N T}}{1-e^{-j2\pi f T}} = \frac{\sin(\pi f N T)}{\sin(\pi f T)}$$

$\rightarrow$ $$\left[X_c(f) = X(f) * \frac{1}{T}\sum_{k} \delta\left(f-\frac{k}{T}\right) = \frac{1}{T}\sum_{k} X\left(f-\frac{k}{T}\right) \right]$$

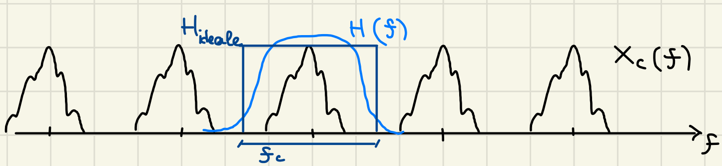


il campionamento di un segnale ne mantiene intatta l'informazione se e solo se $\frac{1}{T} > 2f_{MAX}$ → frequenza di campionamento f_c

$$\boxed{f_c = \frac{1}{T} \geqslant B_x}$$

(x segnali reali $B_x = 2f_{MAX}$)

Applico un filtro per eliminare le "copie" e ricostruire il segnale:



$$X_r(f) = X_c(f) \cdot H(f)$$
$$x_r(t) = x_c(t) * h(t) = \sum_n x_n \cdot h(t - nT)$$

$$H(f)_{ideale} = T\,\mathrm{rect}\left(\frac{f}{f_c}\right) \qquad h(t)_{ideale} = \frac{\sin(\pi t f_c)}{\pi t f_c}$$