Quantum circuits and devices

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Lectures Notes

2022-23

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(Shorter lecture notes may have been explained using slides)

Turing and Von Neumann — 14/09/2023 Lecture 1

Turing → First computer

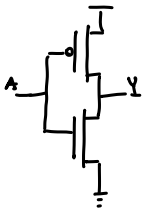
Von Neumann → Developed Von Neumann architecture → CPU + MEMORY (commands, inputs and outputs, storage)

Shannon → father of logic circuits and information theory, first introduced the word "BIT"

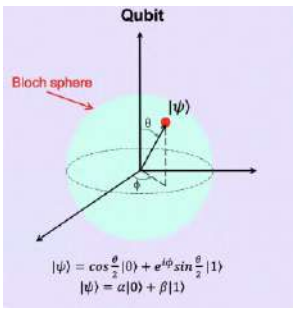
Feynman → simulation of Physics with computer → to simulate nature it is necessary to employ quantum mechanical computation. Classical physics is good only for NON NATURAL events or objects. Natural world can be studied by quantum mechanics.

QUANTUM COMPUTING

When we talk about a bit we have simply a binary digit that can be either 0 or 1 we can implement such a bit with high or low voltages.



The QU-BIT is a quantum bit of information. We still talk about of bits but they are not in a determined 0 or 1 state as before.
We define a function: $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ we are both in 0 and 1 states at the same time. From a geometrical perspective it is like having a bit in a 3D space



We can identify our coordinates also by means of polar ones (θ and ϕ).

α and β are complex numbers but "not free" → they must obey to the constraint $|\alpha|^2 + |\beta|^2 = 1$ (α and β are probabilities).

Examples of qubits:
- Spin state of a single electron (spin up $|z_+\rangle$ or down $|z_-\rangle$)
- Energy state of a confined electron (ground or excited state)
- Direction of the current within a superconducting circuit
- Path of a single photon (path 1 or path 2)
- Polarization of a single photon (horizontal or vertical)

Quantum oscillation, even at the lowest levels of energy, can't STOP!
α and β represent the probability → if I read the SPIN of the QUBIT, I have a certain probability α to be in state 0 and β probability to be in state 1. α and β are defined as PROBABILITY AMPLITUDES.
The phase doesn't represent any info → we want to look at relative phase between α and β. It can be more convenient to look at β and relative phase.
1 and 0 are orthonormal → By making a scalar product we get 0.

DOUBLE QUBIT

Product of the two wavefunctions gives the expression of the double qubit → I can have 4 parameters in this case that describe the probabilities to have the four different states. (00, 01, 10, 11)
By increasing the number of qubits I start to increase exponentially my qubits number.
I have many complex quantities that I can manipulate ($c_0, c_1, c_2 \ldots$)

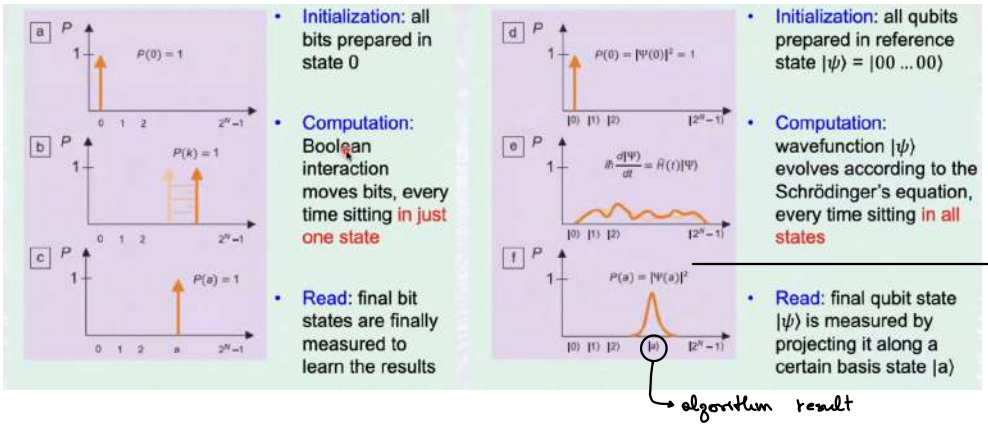
QUANTUM PARALLELISM and INTERFERENCE

By imposing an RF of the right resonance and duration we are able to operate quantum gates. By flipping just one single qubit, I'm operating and moving all the qubits combinations.
$|+\rangle$ → pointing to the equator in the positive part } Hadamard
$|-\rangle$ → pointing to the equator in the negative part }
The interference caused by Hadamard gate makes, in this example to push c_5 both to 111 and 110 → also c_6
c_5 and c_6 become MIXED → Constructive and destructive interference.

ENTANGLEMENT

We saw that we can configure an qubit as superposition of many qubits (00, 01, 10, 11). We can, in a double qubit system, the possibility to have the same state is high. I can get therefore 00 and 11 states. This happens when the two qubits are ENTANGLED → one depends from the other and their superposition

gives only 2 states.



I expect the state to stabilize around a single basis state.

LONG COHERENCE TIME → The state will exist until a perturbation happens. The coherence time gives idea about the time amount in which two qubits remain coherent one to each other. I can perform certain number of operations before the coherence time expires.

QUANTUM VOLUME → The shorter the coherence time, the higher the error rate. But also the precision of the gate highly influences the error rate. By improving the coherence time we can quadratically increase the volume of qubits → we increase both width and lenght.

Quantum SPEED UP -> we can improve our algorithms by applying a parallelism -> the quantum time in order to solve a problem of "search" WRT digital domain is "$\sqrt{N}$" (while digital is "N") -> this is due to the fact that quantum algorithms are parallel.
RSA -2048 -> Number obtained by multiplying 2 prime numbers -> the task is the one to find which are the prime numbers employed (very though task)

Lecture 2 - 19/09/23

QUANTUM MECHANICS

Schrödinger's equation $\Psi(x,t)$ -> unknown (wavefunction)
↳ In such equation I describe a particle with a certain mass, a position and so on. The electron is not described deterministically due to the uncertainty on position and SPIN. We need the position but this element is not given by the Schrödinger equation, what we get is a wavefunction. That's why we ask ourselves: is the electron a particle or a wave?
We introduce Born's postulate that states the probability to have a particle in a certain position in a certain interval. -> $P(x,t)\,dx = |\Psi(x,t)|^2 dx = \Psi^*(x,t)\,\Psi(x,t)\,dx$ ($P(x,t)$ is real). Ψ is also called probability amplitude and it must hold the normalization $\int_{-\infty}^{+\infty} |\Psi(x,t)|^2 dx = 1$. The wavefunction can be both computed analytically or numerically (with errors in this case) but, even if exactly computed, it gives just a probabilistic information.
Heisenberg Uncertainty principle -> $\Delta x\,\Delta p_x \geq \frac{\hbar}{2}$ Δx=measure uncertainty Δp_x=prob. to find particle
For $\Delta x \to 0$ we have $\Delta p_x \to \infty$ in order to preserve the relation. This relation tells us that I may be very precise in calculating the position but I won't have the probability to find the particle in that precise position.
For this reason we introduce the "Expectation value" -> the wavefunction gives us the average position of the particle. In order to estimate the momentum we rely on $\hat{p}_x$, which is an operator.
We cannot use $p_x(x)$ -> it is against quantum mechanics principle: we can't know at the same time both the position and the probability given a certain position to find a particle.
$\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$ -> it is not in function of x but in function of x variation.
Operators -> I use them inside equations in order to obtain "estimated values"
Each OBSERVABLE (physical quantity we want to measure) can be represented by an operator.
At the end, what we find is that Schrödinger's equation can be recognized as a conservation of energy equation.
PENDULUM EXAMPLE: we have a mass attached to a wire of lenght L. When the mass is at the highest point (on right or left) the speed is zero and therefore also the kinetic energy is null ($K=0$) BUT, the potential energy is MAX (because we have gravity) -> $K=0$, V is MAX. When the pendulum passes through the center, in that point we have $V=0$ and $K=MAX$. (Conservation of energy)
The equation behaviour can be observed by means of a "pendulum"
Each operator has its own eigen vectors and eigen values.

$\int_{-\infty}^{+\infty} \psi_i^* \psi_j\,dx = \delta_{ij}$ -> Kronecker's Delta $\begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$
Scalar product of orthonormal eigenfunctions

Commutators -> $[A,B] = \hat{A}\hat{B} - \hat{B}\hat{A}$

$$\frac{\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)}{\psi(x)} = E$$

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right]\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,t)$$

Lecture 3 – 21/09/23

$\hat{H}$ → Hamiltonian

If $V(x,t) = V(x)$ we can split the equation relatively to time and space.

$\Psi(x,t) = \Psi_u(x)\cdot e^{-i\omega_u t}$ BASIS STATE or STATIONARY STATE

↳ u is the quantum number and it is a discrete number $(1,2,3\ldots)$

ω_u is linked to the eigenvalue of energy → $\omega_u = \frac{E_u}{\hbar}$

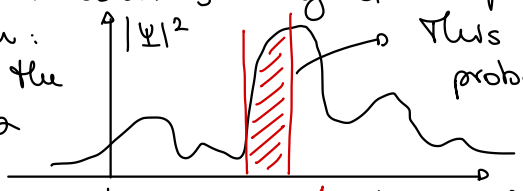
$\hat{H}\Psi_u = E_u\Psi_u$ (the highlighted relation is obtained from this equation.

↳ the Hamiltonian represents the total energy of the system.

$\Psi(x,t) = \sum_u C_u \Psi_u(x) e^{-i\omega_u t}$ SUPERPOSITION STATE (contains all the basis states at the same time). This is NO MORE STATIONARY.

We defined $p(x)\,dx = |\Psi|^2\,dx$ → Probability to find the particle in a portion dx

Taken the wavefunction spectrum: In the case of stationary state, the probability to find a particle in a certain interval is time invariant.



This area represents the probability to have a particle in the space $[x, x+dx]$

In the case of SUPERPOSITION this condition doesn't hold anymore →

→ it is due to the summation of all the different phases that are time dependent.

FREE PARTICLE EXAMPLE

Not only the potential depends only space but it is constant → no field applied to the particle and therefore no gradient of the potential since no force is applied.

↳ the momentum remains the same.

$V(x,t) = V(x) = V = \text{const.}$

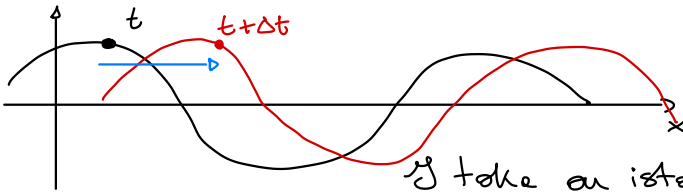
$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \quad E>0$

By using the Eulerian expression it results easier to compute. (in other occasions we may opt for sine and cosine)

$\Psi(x) = Ae^{ikx} + Be^{-ikx}$ PHASE: $\varphi(t) = \exp\left(-i\frac{E}{\hbar}t\right)$

$\Psi(x,t) = Ae^{i(kx-\omega t)} + Be^{-i(kx+\omega t)}$ ↳ we include the t exponent arg.

↳ We see that the wavefunction consists of 2 moving waves.



By taking the real part of the written expression we'll have a cosine expressed as follows: $\cos(kx-\omega t) = 0$

I take an istant t and $t+\Delta t$. The overall phase is $(kx-\omega t)$

If I want to see how the phase zero moves according to time I get:

$kx - \omega t = 0 \quad x = \frac{\omega}{k}t = v_\varphi t$ → phase velocity. So we have a linear dependence on time for what concerns the phase according to the found relation.

The velocity is positive and therefore we have a variation in the direction highlighted in blue. We call this wave TRAVELING WAVE. NOTE: The particle remains a particle but it can be described by means of waves and probabilistic relations.

The second term also represents a wave: $\Psi(x,t) = Ae^{i(kx-\omega t)} + Be^{-i(kx+\omega t)}$

How do we retrieve information from a wavefunction?

I take the square modulus: (by considering just first term) we notice that $Ae^{i(kx-\omega t)}$ is a pure phasor with constant modulus A.

Therefore by computing $|\Psi|^2 = |A|^2$ we obtain a constant → constant probability density to find a particle → NO information about WHERE (position)

↳ This wave moves in the opposite direction WRT the first wave.

we have that $\Delta x = \infty$ → the uncertainty principle of Heisenberg tells that $\Delta p\Delta x \geq \frac{\hbar}{2} \Rightarrow \Delta p = 0$

we get a very precise momentum $p = \hbar k$ → this comes from the fact that: If $\Delta p \to 0$ then $\Delta p = 0$ in order to have finite result

$-i\frac{\hbar}{\partial x}\psi = p_x\psi$ → eigenfunction of the momentum operator.

$-i\frac{\hbar}{\partial x}\psi = p_x\psi$ momentum operator

↓ we substitute e^{ikx}

$-i\hbar\, ik\,\psi = p_x\psi$

↓ we obtain the previous result

$p_x = \hbar k$

By substituting ψ in the Schrödinger equation we can take the expressions for E and k

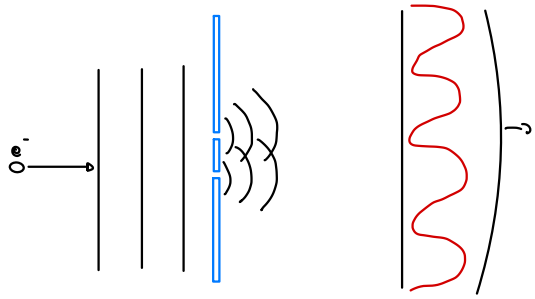
$E = \frac{\hbar^2 k^2}{2m} = \frac{p^2}{2m}$ (since there's no applied force, the only energy is the kinetic one.

$k = \frac{\sqrt{2mE}}{\hbar}$ → Definition of kinetic energy

$\omega = \frac{E}{\hbar}$ (link of ω to E)

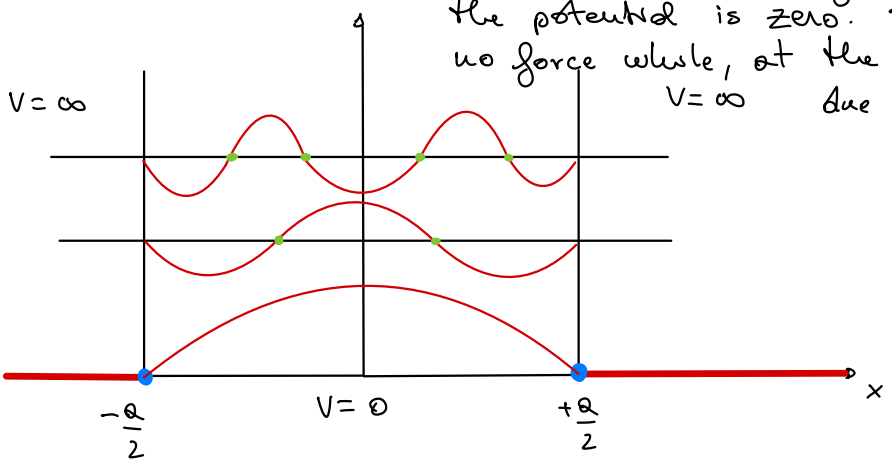
Eigenvalue of energy is kinetic one.

EXPERIMENT: I assume to have a source of electrons and I put a screen with two small "slits". We can observe an interference.



The electron remains a particle BUT its behaviour is described by means of waves.
The interference pattern tells us the probability of the electron to hit the cathode in a certain position.
The wavefunction describes the probability of the particles to be in a certain position.

INFINITE WELL PROBLEM: We have a region confined in a space large "a" where the potential is zero. In this region the electron sees no force while, at the boundaries, it sees an infinite force due to the infinite gradient.



$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$

↓

At the boundary, in order to satisfy the equation, if $V=\infty$ then ψ must be zero → $\psi = 0$ in order to get a finite expression.

Now the problem is the one to solve the differential equation by imposing an boundary conditions.

$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$ ← we may write ψ in Eulerian form but in this case I'm not expecting a travelling wave. Instead, I'm expecting a wave continuously reflected back and forth at the boundaries and stationary. The wave is expected to be confined in $-\frac{a}{2}$ and $\frac{a}{2}$ (blue dots).

In the middle I may expect a cosine.

Since I expect a stationary wave, the green dots in the sketch are expected to stay fixed (the points in which the wavefunction is nil)

We introduce even and odd eigenfunctions:

EVEN $\psi(-x) = \psi(x)$ (COSINE)

$\psi(x) = A\cos(kx)$ → $k = \frac{\sqrt{2mE}}{\hbar}$ (link with energy). In the previous case E could have been any value but, in this situation we have some boundary condition.

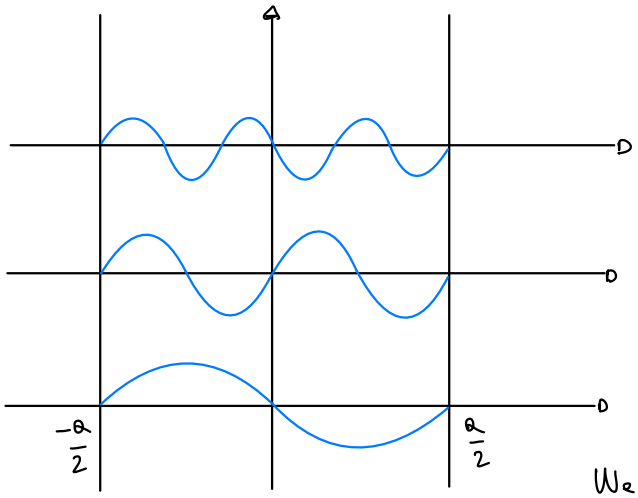
$a = n\frac{\lambda}{2} = \frac{n}{2}\frac{2\pi}{k}$ ← $a = n\frac{\lambda}{2}$ λ is the full wave, since we get only half wave we impose $\frac{\lambda}{2}$

λ is the wavelenght but we are interested in k that is the wave number $\lambda = \frac{2\pi}{k}$

We substitute and obtain the boundary condition of k → $k = n\frac{\pi}{a}$ with $n = ODD = 1, 3, 5 \ldots$

ODD $\psi(-x) = -\psi(x)$ (SINE) If I flip the axis I see the same function but with a minus.

Also in this case we have wavefunc. equal to zero in $\frac{a}{2}$ and $-\frac{a}{2}$ and we have an EVEN number of half-waves.



$\Psi = B \sin kx$

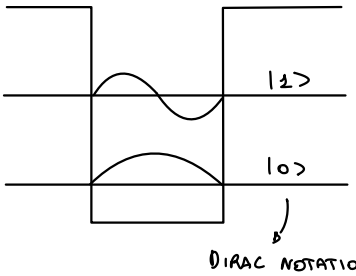
$a = n \frac{\lambda}{2}$ $\quad n = \text{EVEN} = 2, 4, 6 \ldots$

The k formula is equal also for ODD functions but n is an even number.

$E_n = \frac{\hbar^2 k^2}{2m} = \frac{h^2}{8ma^2} n^2$ The eigenvalue of the electric field increases with the order n.

$\hbar = \frac{h}{2\pi}$

We are close to the qubit situation:



1- Bosys state

0- Bosys state

DIRAC NOTATION

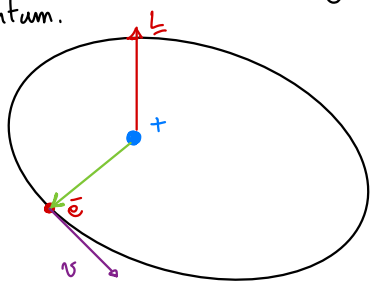
Used to express the bosys state of a particle.

We can have also a state |2> but we generally prefer to avoid it and have only |0> and |1> → obviously this is due to the necessity to have a binary driven information. The |0> state represents the "GROUND STATE" while the |1> state is the EXCITED STATE.

SPIN

It is an important class of Q-bit. First of all we introduce the angular momentum.

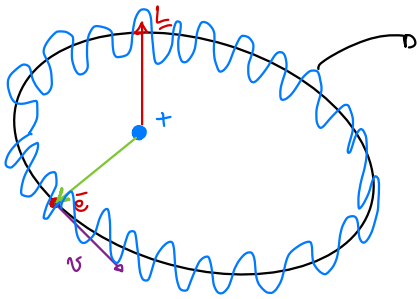
$\underline{L} = \underline{r} \times \underline{p}$



We have an electron moving on an orbit with radius r WRT the center of the system, which is a proton in this case, with a velocity v.

The angular momentum is vectorial product of $\underline{r}$ and $\underline{p} = m \cdot \underline{v}$

In a classical situation we have that a mass can orbit around another body with a constant angular momentum if a force exists.
Now we observe what happens in the QUANTUM world. The particle in this case can be either traveling as shown but, it behaves like a wave.



The electron moves like a waves and always come back to the same point. This traveling wave can interfere with itself. Generally this wave is always destructively interfering.
$|\Psi|^2 = 0$ (For destructive interference)
There could be a condition for which the electron is in phase with itself and in that case $|\Psi|^2 \neq 0$ → CONSTRUCTIVE INTERFERENCE → considering a circular path we have that:

$2\pi r = n\lambda = n \frac{2\pi}{k} \cdot \frac{h}{p} = n \frac{h}{p}$ $\quad (\hbar = 2\pi\hbar, \; p = \hbar k)$

↳ DE BROGLIE wavelenght

$L = pr = n \cdot \frac{h}{2\pi} = n\hbar$ → The momentum is quantized $L = n \cdot \hbar$

We have 2 types of quantization:

- $L^2 = \hbar^2 l(l+1)$ l = orbital quantum number = 0, 1, 2 ... (orbitals: s p d)
  ↳ modulus of the angular momentum vector
- $L_z = \hbar m$ $\quad m = 0, \pm 1, \pm 2, \ldots, \pm l$
  ↳ quantization along a preferred axis → we can have either positive or negative quantum number since the projection can be both + or −

This means that not every orbital is good in order to have a particle since the momentum must be a multiple of $\hbar$.

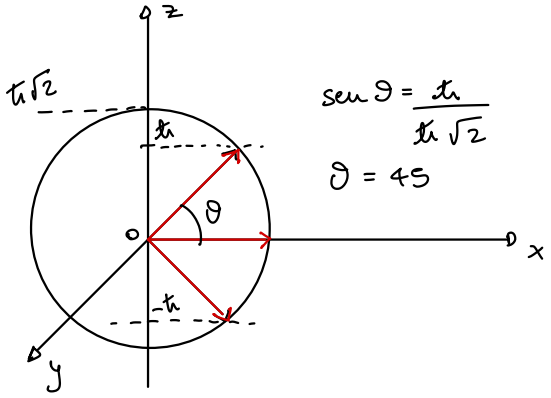
For $l=0 \rightarrow m$ can only be $0 \rightarrow m=0$. In the ground state of hydrogen atom we don't have any angular moment → How do particles orbit? The particle should move only radially WRT the radius of the orbit.

$l=0 \quad m=0 \rightarrow L=0$

$L_z=0$

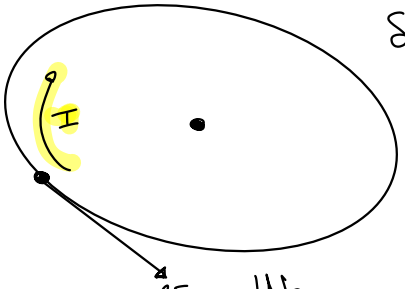
$l=1 \quad m=0, \pm 1 \rightarrow L=\hbar\sqrt{2}$

$L_z=0, \pm\hbar$



$\operatorname{sen}\vartheta = \frac{\hbar}{\hbar\sqrt{2}}$

$\vartheta = 45$



Since I have a charge moving along an orbit, then it will be like having a current flowing in the opposite direction.

We can evaluate the magnetic moment:

$\mu = I \cdot A = \frac{q}{2\pi r} v \cdot \frac{m}{m} \cdot \pi \cdot r^2 = L_z \cdot \frac{q}{2m}$

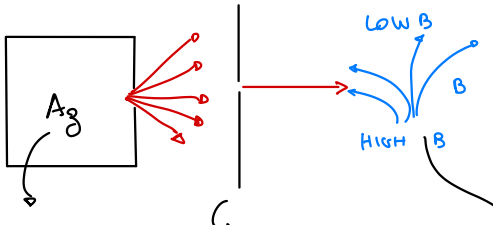
$I = \frac{q}{2\pi r} v$

$A = \pi r^2$

$q<0$

$\mu_z = \frac{q}{2m} L_z$

STERN-GERLACH EXPERIMENT → To measure the magnetic dipole moment of the single particles



$V = -\underline{\mu} \cdot \underline{B} = -\mu_z B_z$

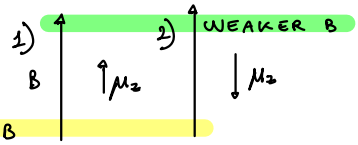
↳ Potential energy

Vaporized silver inside a chamber

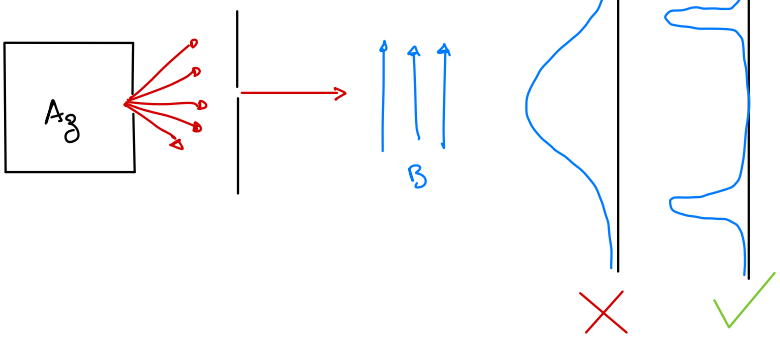
↳ This SLIT focuses the beam of particles

Application of a NON UNIFORM magnetic field → More precisely a SPREADING B, decreasing from bottom to the TOP

We can have two possibilities:



In the situation 1 the dipole tends to align with the magnetic field and flips, resulting a lower potential. In the situation 2 the antiparallel μ_z tends to align upwards towards the lower B.



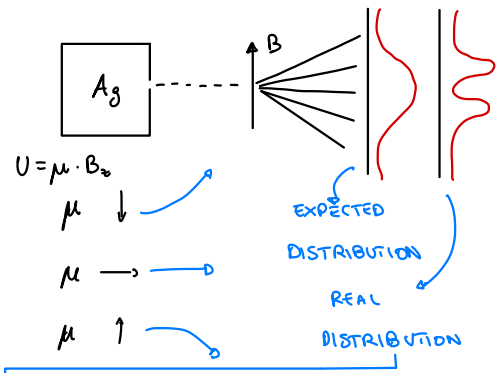
Stern and Gerlach where expecting a wide distribution as the one in figure 1. What they discovered was a situation like the one in point 2, well focused on two points.

↓

DISCOVERY OF THE SPIN

Lecture 4 - 26/09/23

STERN-GERLACH EXPERIMENT → Oven with a little hole to let heated silver particles pass through. These particles are sent towards a magnetic field B, pointing up with a gradient (more intense toward up direction WRT the bottom).



If we have any magnetic dipole pointing up, this would like to bend down in order to stay parallel WRT the B_z field → the larger the B field, the larger the potential energy. We have many magnetic dipoles and according to where they are pointing at, they will bend toward a certain direction. What was expected is a gaussian like distribution of the particles on the screen.

↳ The experiment revealed instead that the particles were distributed in two precise positions, thus revealing that the magnetic dipole is QUANTIZED.

We have a contraddiction: L is quantized and it is related to μ as follows: $\mu_z = \frac{q}{2m} L_z$ → ↳ angular momentum

→ L_z is always ODD,

Therefore we have that for the different orbitals: s ($l=0$) $L_z = 0$ so we have that the quantization cannot be due to angular momentum p ($l=1$) $L_z = 0, \pm 1$

The reason is: looking at silver atomic config d ($l=2$) $L_z = 0, \pm 1, \pm 2$

we have 47 electrons. 46 of these complete the orbitals → they complete s, p, d levels and so we have no angular momentum since for each electron with $L_z = +2$ there's another with opposite spin that balances. Then there is the last electron that is at the level s → $L_z = 0$ → NO ANGULAR MOMENTUM → → Silver electrons doesn't contribute to the angular momentum. It comes out that what is contributing to the distribution seen on the screen is the SPIN of this last electron. The SPIN consists of an intrinsic angular momentum given by the rotation of the electron on it self. This SPIN follows the same behaviour of the angular momentum and it is quantized.

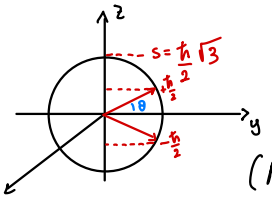
$S^2 = \hbar^2 s(s+1)$

$S_z = \hbar s_z$ $\quad s_z = -s, -s+1 \dots s-1, s$ (as angular momentum)

↳ SPIN projected along z axis ↳ As we see from the screen it must be equal to 2 → therefore s can't be equal to an INTEGER number. We have to assume s as semi-integer We can define two families:

BOSONS → characterized by an integer number of s → $s = 1$ (PHOTONS), $s = 0$ (α PARTICLE, He^{2+} nucleons)

FERMIANS → semi-integer s → $s = \frac{1}{2}$ (for electrons) → $s_z = -\frac{1}{2}, +\frac{1}{2}$



$\sin\theta = \frac{\hbar/2}{\hbar/2\sqrt{3}} = \frac{1}{\sqrt{3}}$

This explains the STERN-GERLACH experiment result: Silver has a magnetic dipole moment which can be attributed to the SPIN of the electron in the s orbital. At this point we might argue that Silver has a nucleons that also contributes to the magnetic dipole. (Also protons and neutrons are FERMIANS with SPIN $\pm \frac{1}{2}$). The reason why the nucleons can be considered negligible is found in the expression of $\mu_z = \frac{q}{2m} L_z$ (L_z will be replaced with s_z) As we see the magnetic dipole goes as $\frac{1}{m}$ → the mass of the protons and neutrons is much larger WRT the one of electrons. $m_{electron} = 9{,}1 \cdot 10^{-31}$ kg $- m_{neutron,proton} = 1831\, m_{electron}$ → the magnetic dipole moment related to the nucleons has a much lower impact WRT the electron one.

$\mu_z = g \frac{q}{2m} S_z$ ← g-factor = 2,0023 (for the electron) → the charge of the electron is not uniformly distributed all over the "sphere"  , the g-factor accounts for such distribution of the electron charge along the trajectory of the spinning electron. The g accounts for the larger effective charge in the periphery of the sphere.

$\mu_z \simeq \frac{q}{m} S_z$ → 2 and g can be approximately simplified

$\gamma = g \frac{q}{2m}$ → $\mu_z = \gamma S_z$ ← GYROMAGNETIC RATIO

$\mu_z = \pm g \frac{q}{2m} \cdot \frac{\hbar}{2} = \pm \mu_B$ → BOHR MAGNETIC DIPOLE MOMENT OR BOHR MAGNETON

$\mu_B = 9{,}27 \cdot 10^{-24}\ J/T$ can be used also to calculate energy

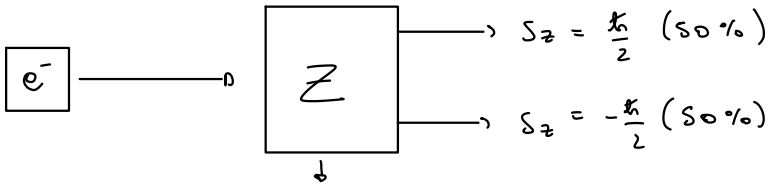
Also neutron and proton have its own g-factor. Although neutral, neutrons show a magnetic dipole moment. The neutral charge can be seen as a distribution of + and - charges with the negative charges pushed toward

the external part, resulting in a sort of negative charge. Another fact regarding the SPIN regards the electron radius: we can estimate it assuming that the electron moves all around a sphere and that the energy of the electron is an electrostatic one

1) $\frac{\hbar}{2} = m \cdot v \cdot r$ (intrinsic angular momentum) 2) $E = \frac{q^2}{4\pi\varepsilon r} = mc^2 = 0{,}511\,MeV$ (relativistic energy) → from this relation we extract $r \cong 10^{-15}\,m$ → if we substitute in equation (1) we find $v \cong 10^{10}\,m/s$ → NOT POSSIBLE SINCE it is larger than speed of light.

The paradox is solved with the assumption that the SPINNING is just a property of the particle, like mass and charge. We take it as it is, we can't find a model to understand the effective trajectory and speed during the spinning.

We now assume to have a source of electrons (an electron gas for example)



STERN GERLACH APPARATUS → This will separate the electrons depending on the spin

I'm able to split the gas beam in two beams composed by electrons either with SPIN UP or DOWN. The SPIN is a wave-function, I can imagine it as a measurable property of the electron and I can describe it with an operator approach:

$\hat{O}|\psi\rangle = o|\psi\rangle$ (operator eigenfunction) → $S_z = \pm\frac{\hbar}{2}$ (angular momentum along z-axis - SPIN) → In this case we talk about eigenvalue

(in this case it is the spin operator)

I can assume that my electrons are in a superposition state:

$|\psi\rangle = \alpha|z_+\rangle + \beta|z_-\rangle$

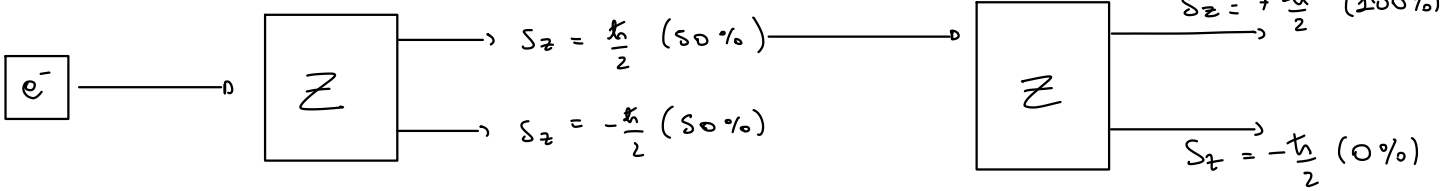
eigenfunction of the positive part of z operator $|z_+\rangle$ → $S_z = +\frac{\hbar}{2}$ ($|z_-\rangle \rightarrow S_z = -\frac{\hbar}{2}$)

→ So we can replace the previous general formula and write: $\hat{S}_z|z_+\rangle = \frac{\hbar}{2}|z_+\rangle$ $\hat{S}_z|z_-\rangle = -\frac{\hbar}{2}|z_-\rangle$

$|\psi\rangle = \alpha|z_+\rangle + \beta|z_-\rangle$ → this is not a description from the statistical point of view of how many electrons with SPIN UP or SPIN DOWN are present inside the chamber. Each electron has both SPIN UP and SPIN DOWN. $|\psi\rangle$ is the quantum mechanical description of the electron and it must be taken as it is. $|z_+\rangle$ and $|z_-\rangle$ are the basis states and the measurement performed by means of the z chamber is to FORCE (and destroy) the superposition state in one or another. In a quantum computer when we perform a readout we DESTROY the computation, we can't get back to previous steps.

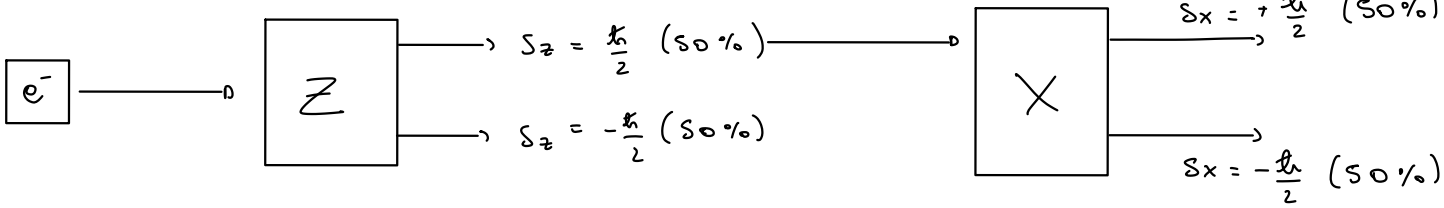
NOTATION: KET $|\cdot\rangle$ BRA $\langle\cdot|$

$|\alpha|^2 + |\beta|^2 = 1$ → α and β represent the probabilities $\alpha = \beta = \frac{1}{\sqrt{2}}$ → satisfies the same probability to have both the spins UP or DOWN after z.



We now assume to perform another measurement along z-axis In the particular case the probability to measure a SPIN UP is 100% since all the electrons entering the chamber are already in the basis state $|z_+\rangle$.

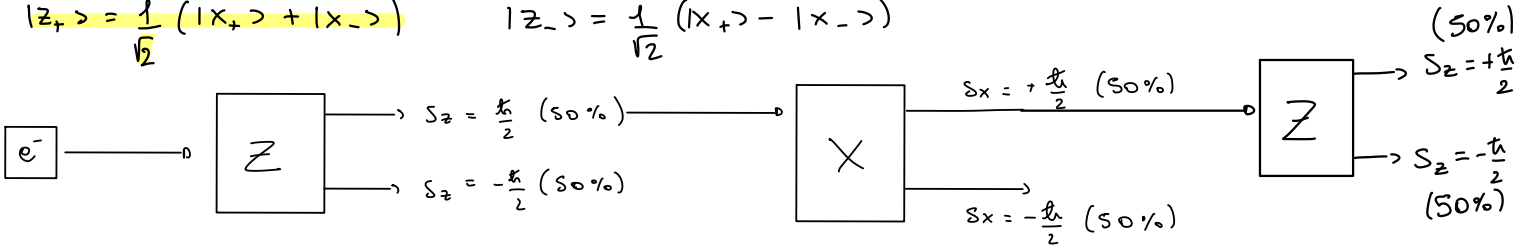
But now we perform a second measurement along a different axis:



As we see the eigenvalues remain the same but the eigenfunction changes: we are using a different operator and different basys state. From this measurement I will end up again with 50% probability to have spin up and 50% for spin down the basys state of the operator $\hat{z}$ is not COMMUTING with the basys state of the operator $\hat{x}$ (they are not showing the same basys state).

$|z_+\rangle$ state can be written as a superposition state composed by $\hat{x}$ operator basys states:

$$|z_+\rangle = \frac{1}{\sqrt{2}}(|x_+\rangle + |x_-\rangle) \qquad |z_-\rangle = \frac{1}{\sqrt{2}}(|x_+\rangle - |x_-\rangle)$$



We now send one of the output basys states of x toward another Stern-Gerlach box, in order to perform another measurement along the z-axis. On the contrary of what was expected, the measurement gives again the same probability to get $|z_+\rangle$ or $|z_-\rangle$ states at the output of the z-box. In fact, the x basys state can be written as a superposition state of z-basys states.

I can perform 3 measurements along the different axis → depending on this I define the different operators that express what we have seen in the schematics above:

$$\hat{S}_x|x_+\rangle = +\frac{\hbar}{2}|x_+\rangle \qquad \hat{S}_x|x_-\rangle = -\frac{\hbar}{2}|x_-\rangle$$
$$\hat{S}_y|y_+\rangle = +\frac{\hbar}{2}|y_+\rangle \qquad \hat{S}_y|y_-\rangle = -\frac{\hbar}{2}|y_-\rangle$$
$$\hat{S}_z|z_+\rangle = +\frac{\hbar}{2}|z_+\rangle \qquad \hat{S}_z|z_-\rangle = -\frac{\hbar}{2}|z_-\rangle$$

We express our states by means of matrixes $|z_+\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $|z_-\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ → What are representing these columns? They are the α and β values → $|\psi\rangle = \alpha|z_+\rangle + \beta|z_-\rangle = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$

What is $\hat{S}_z$? It must be something that multiplied by a vector it gives me back a vector: A MATRIX. by substituting $|z_+\rangle$ and $|z_-\rangle$

$$\hat{S}_z = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \longrightarrow \begin{cases}\hat{S}_z\begin{pmatrix}1\\0\end{pmatrix} = +\frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix}\\ \hat{S}_z\begin{pmatrix}0\\1\end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix}\end{cases}$$

I have to find a matrix $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ whose eigenvectors are $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$. From this we obtain that:

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

All the operators have the same shape

For x and y operators we have to remember: $|x_+\rangle = \frac{1}{\sqrt{2}}(|z_+\rangle + |z_-\rangle)$ and $|x_-\rangle = \frac{1}{\sqrt{2}}(|z_+\rangle - |z_-\rangle)$ by replacing the vectors representing $|z_+\rangle$ and $|z_-\rangle$ we get that $|x_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $|x_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ → I have the eigenvectors and I can write the equations in order to compute the matrix of $\hat{S}_x$ operator: $\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

In the end we have y axis whose basys state can be written as superposition of $|z_+\rangle$ and $|z_-\rangle$ states as follows: $|y_+\rangle = \frac{1}{\sqrt{2}}(|z_+\rangle + i|z_-\rangle)$ $|y_-\rangle = \frac{1}{\sqrt{2}}(|z_+\rangle - i|z_-\rangle)$ (we start to see the complex number nature of α and β). By repeating the same substitutions applied before we get $\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$ The operators we got are the SPIN OPERATORS → from these ones we can get the PAULI OPERATORS = SPIN OPERATORS / $\hbar/2$ such operators can be used as basys states for matrixes: I need another basys state since we may need a fourth element for our matrix. Here comes the identity matrix to work as basys state operator: $\hat{\sigma}_I = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ and at this point we may express our matrix $\hat{A} = \alpha\hat{x} + \beta\hat{y} + \gamma\hat{z} + \delta\hat{I}$

PAULI OPERATORS

$$\hat{\sigma}_x = X = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad \hat{\sigma}_y = Y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \quad \hat{\sigma}_z = Z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

NOTE THAT:

$$X^2 = I \qquad XY = iZ$$
$$Y^2 = I \qquad YZ = iX$$
$$Z^2 = I \qquad ZX = iY$$

Lecture 5 – 28/09/23

PAULI OPERATORS

$\hat{\sigma}_x = \hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\hat{\sigma}_y = \hat{Y} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$

$\hat{\sigma}_z = \hat{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

The difference between PAULI OPERATORS and SPIN OPERATORS is just $\frac{\hbar}{2}$. They have the same eigenfunction but different eigenvalues. (FOR SPIN OP. → $\pm\frac{\hbar}{2}$ while for PAULI OP. → ± 1)

$\left(\hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z\right)$

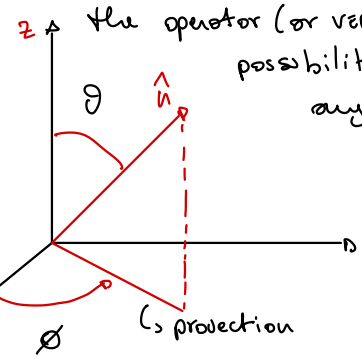
When dealing with single SPIN OPERATOR qubit the matrixes are just 2×2.

$|z_+\rangle = |0\rangle$ } Bosys state of our qubit

$|z_-\rangle = |1\rangle$ }

Also $|x_+\rangle$, $|x_-\rangle$, $|y_+\rangle$ and $|y_-\rangle$ are bosys states but they are defined on the bosys of $|z_+\rangle$ and $|z_-\rangle$.

HOW ABOUT ANY GENERIC AXIS? We introduce the operator (or VERSOR) $\hat{n}$ that give us the possibility to evaluate the SPIN along any direction different from x,y,z



$n_x = \text{sen}\,\vartheta\cos\phi$

$n_y = \text{sen}\,\vartheta\,\text{sen}\,\phi$

$n_z = \cos\vartheta$

ϑ → angle of $\hat{n}$ WRT z axis

ϕ → angle of the projection of n on the x,y plane (we will eventually define it in terms of phase) WRT x axis.

$$\hat{\sigma}_n = \text{sen}\,\vartheta\cos\phi\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \text{sen}\,\vartheta\,\text{sen}\,\phi\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \cos\vartheta\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =$$

($\hat{\sigma}_x$, $\hat{\sigma}_y$, $\hat{\sigma}_z$)

$$= \begin{bmatrix} \cos\vartheta & \text{sen}\,\vartheta(\cos\phi - i\cos\phi) \\ \text{sen}\,\vartheta(\cos\phi + i\,\text{sen}\,\phi) & -\cos\vartheta \end{bmatrix} = \begin{bmatrix} \cos\vartheta & \text{sen}\,\vartheta\, e^{-i\phi} \\ \text{sen}\,\vartheta\, e^{i\phi} & -\cos\vartheta \end{bmatrix}$$

($e^{-i\phi}$; $e^{i\phi}$)

I expect that $e^{i\phi}$ the eigenvalues are the same as for the PAULI OPERATORS. To get them we have to find $\det(\hat{\sigma}_n - \lambda I) = 0$

$$\det\begin{bmatrix} \cos\vartheta - \lambda & \text{sen}\,\vartheta\, e^{-i\phi} \\ \text{sen}\,\vartheta\, e^{i\phi} & -\cos\vartheta - \lambda \end{bmatrix} = 0 \rightarrow -\cos^2\vartheta - \lambda^2 - \text{sen}^2\vartheta = 0 \quad \lambda^2 = 1 \quad \lambda = \pm 1$$

→ eigenvalues equal to PAULI OPERATORS ones!

In order to get the eigenvectors it is necessary to do some steps more.

Let's assume to start from $\lambda = 1$. I must have that the previous matrix, multiplied by my eigenvector gives me zero.

$$\begin{pmatrix} \cos\vartheta - 1 & \text{sen}\,\vartheta\, e^{-i\phi} \\ \text{sen}\,\vartheta\, e^{i\phi} & -\cos\vartheta - 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

I get two equal equations. One equation can be derived by linear combination of the other equation. I take the second equation in this case.

$\text{sen}\,\vartheta\, e^{i\phi}\, u_1 = (\cos\vartheta + 1)\, u_2$

$$\frac{u_1}{u_2} = \frac{\cos\vartheta + 1}{\text{sen}\,\vartheta\, e^{i\phi}}$$

$\cos\vartheta + 1 = 2\cos^2\frac{\vartheta}{2}$

$\text{sen}\,\vartheta = 2\,\text{sen}\frac{\vartheta}{2}\cos\frac{\vartheta}{2}$

$$\frac{u_1}{u_2} = \frac{2\cos^2\frac{\vartheta}{2}}{2\,\text{sen}\frac{\vartheta}{2}\cos\frac{\vartheta}{2}\, e^{i\phi}} = \frac{\cos\frac{\vartheta}{2}}{\text{sen}\frac{\vartheta}{2}\, e^{i\phi}}$$

The first eigenvector that I get is $|u_+\rangle = \begin{pmatrix} \cos\frac{\vartheta}{2} \\ \text{sen}\frac{\vartheta}{2}\, e^{i\phi} \end{pmatrix}$ the second one is obtained by putting $\lambda = -1$ and we get $|u_-\rangle = \begin{pmatrix} \text{sen}\frac{\vartheta}{2} \\ -\cos\frac{\vartheta}{2}\, e^{i\phi} \end{pmatrix}$

NOTE: $\begin{pmatrix} |\psi\rangle = \alpha|0\rangle + \\ \beta|1\rangle \end{pmatrix}$

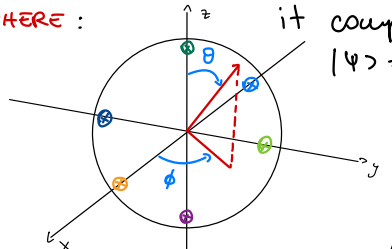
Such α and β are supposed to be normalized:

$|\alpha|^2 + |\beta|^2 = \cos^2\frac{\vartheta}{2} + \text{sen}^2\frac{\vartheta}{2} = 1$ → we checked it

→ These results consist of the states of my blocksphere pointing toward the positive part or the negative one

Instead of using α and β we can use ϑ and ϕ which give more information about WHERE, in the xyz plane, is my SUPERPOSITION STATE.

I can therefore introduce the BLOCKSPHERE:



it comprises all the possible positions of my superposition state.

$|\psi\rangle = \begin{pmatrix} \cos\frac{\vartheta}{2} \\ \text{sen}\frac{\vartheta}{2}\, e^{i\phi} \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

↳ these are not the coordinates of the point, but for any qubit state I can see where it is located in my blocksphere.

Pauli operators represent specific points of my blocksphere:

$|z_+\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|z_-\rangle = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$|x_+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

By substituting the different α and β we find all the bosys states positions.

$|x_-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$ ⑥

$|y_+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ i\frac{1}{\sqrt{2}} \end{pmatrix}$ ④

$\theta = \frac{\pi}{2}$ $\phi = \frac{\pi}{2}$ while for $|y_-\rangle$ ⑧ we get $\theta = \frac{\pi}{2}$ and $\phi = \frac{3}{2}\pi$

NOTE: The block sphere representation can be misleading since we would think that it offers a precise representation of the different states. For instance $|z_+\rangle$ and $|z_-\rangle$ are orthonormal between each other, this means that there's no projection of one vector onto the other and thus, the scalar (or inner) product between the two gives back 0. From the blocksphere instead we may be induced in thinking that the two are actually PARALLEL BUT it is not like that We must not represent the orthonormality in the blocksphere but we don't have to confuse it as it would have been parallel.

QUBIT

When we deal with qubit we can write our state as a superposition of bosys states: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ but when it comes to two qubits, we have to write a composite state:

$|\psi\rangle = (\alpha_1|0\rangle_1 + \beta_1|1\rangle_1)(\alpha_2|0\rangle_2 + \beta_2|0\rangle_2) = \underbrace{\alpha_1\alpha_2}_{\alpha_{00}}\underbrace{|0\rangle_1|0\rangle_2}_{|00\rangle} + \underbrace{\alpha_1\beta_2}_{\alpha_{01}}\underbrace{|0\rangle_1|1\rangle_2}_{|01\rangle} + \underbrace{\beta_1\alpha_2}_{\alpha_{10}}\underbrace{|1\rangle_1|0\rangle_2}_{|10\rangle} + \underbrace{\beta_1\beta_2}_{\alpha_{11}}\underbrace{|1\rangle_1|1\rangle_2}_{|11\rangle} = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$

→ Such two qubit state cannot be represented on the block sphere, it is good only for 1 qubit representation.

By doing a measure on this state we have a probability both first and to have each different second qubit one bosys state (composite) in bosys state two.

$|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$

These are probabilities

If I have any number of qubits, for instance $n=3$, then the number of possible states is 2^n:

$\begin{pmatrix} \alpha_{000} \\ \alpha_{001} \\ \vdots \\ \alpha_{110} \\ \alpha_{111} \end{pmatrix}$

we have that by changing one single state, we modify also the other qubits. Therefore in quantum computers we can operate in a PARALLEL way with all the available qubits → STRENGTH OF QUANTUM COMPUTERS.

DIRAC NOTATION

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ KET

↳ Representation of a wave function

$\langle\psi| = [(|\psi\rangle)^T]^*$ BRA is the KET CONJUGATE TRANSPOSE

$\langle\psi| = |\psi\rangle^t$

EXAMPLE: $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \rightarrow \langle\psi| = (\alpha^* \ \beta^*)$

BRAKET: $\langle\psi|\psi\rangle$ gives a scalar

↳ It consists of the INNER PRODUCT. If we compute the inner product of two bosys state we get a KRONEKER'S DELTA

HADAMARD BASYS STATES:

$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

ORTHONORMAL

$\langle\psi_i|\psi_j\rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

$\begin{cases} \langle\psi|\psi\rangle = (\alpha^* \ \beta^*)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = |\alpha|^2 + |\beta|^2 = 1 \\ \langle 0|0\rangle = (1 \ 0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \\ \langle 0|1\rangle = (1 \ 0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \end{cases}$

As we have inner product we also have the OUTER PRODUCT: this gives a matrix

$|\psi_1\rangle\langle\psi_2| = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \otimes (\alpha_2^* \ \beta_2^*) = \begin{pmatrix} \alpha_1\alpha_2^* & \alpha_1\beta_2^* \\ \beta_1\alpha_2^* & \beta_2\alpha_2^* \end{pmatrix}$ → I get an operator (or MATRIX)

⊗ tensor product

EXAMPLE: write $\hat{x}$ as summation of OUTERPRODUCTS $\hat{x} = |1\rangle\langle 0| + |0\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix}(1 \ 0) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}(0 \ 1) =$

$= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ Verified ✓

2-QUBIT STATE

Let's make an example of composite state:

$|\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

$|\psi_2\rangle = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$

the two qubit state can be obtained by the tensorial product of the two states (we get a 4×1 matrix from two 2×1).

$|\psi\rangle = \frac{1}{2\sqrt{2}}\begin{pmatrix} \sqrt{3} \\ 1 \\ -i\sqrt{3} \\ -i \end{pmatrix}$ → Verify that also this eigenvector is normalized:

$\frac{1}{8}(3 + 1 + 3 + 1) = 1$ Verified ✓

The different probabilities $(\alpha_{00}, \alpha_{01}\ldots)$ can be found from the initial probabilities (which are independent among them):

$(1,1) \leftarrow \frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$ → probability of being in $|00\rangle$ state

↳ $(2,1)$ (remember that the probabilities are $|\alpha_i|^2$)

ENTAGLED STATE

$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ → This is not a composite state since I cannot write it in terms of basys states. In ENTANGLED STATE I have 50% to be in state $|0\rangle$ and 50% to be in state $|1\rangle$. In this case if I measure the second qubit I will be sure at 100% that it is in $|0\rangle$ state in one case or $|1\rangle$ in the other → it is ENTANGLED, there's no more independency. It is more usual to have an ENTANGLED STATE more than a COMPOSITE one. A special ENTANGLED STATE are the BELL STATE:

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\Phi_-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\Psi_+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

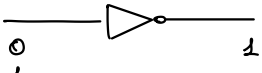
$$|\Psi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

EPR – Einstein Podolski Rosenberg paradox

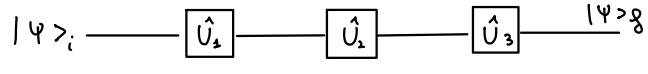
Assuming two have two electrons prepared in $|\Phi_+\rangle$, one gets absorbed by a point A and the other in B but they are still entangled, until one of the two is subject to a measurement. We have 50% possibility to be in state $|0\rangle$ or $|1\rangle$. We assume to measure $|0\rangle$. What happens is that the result info is IMMEDIATELY transmitted to the other point. This is IMPOSSIBLE for the relativity principle that states anything can travel faster than light. In reality, what happens is something completely random and intrinsic of nature.

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\theta}\sin\frac{\theta}{2}|1\rangle$

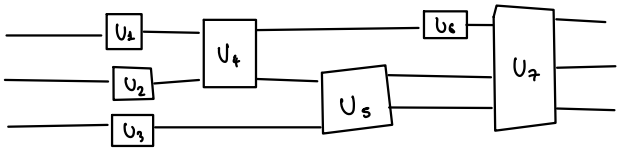
Lecture 6 - 10/10/23

Today we see the QUANTUM GATES → How do we manipulate such quantum bits?
A simple example is the 1 bit inverter: 

this is not a problem in quantum computing. In quantum computing the use of more input bits becomes difficult → we cannot have a number of output bits different from the input ones. More over, in classic digital world we have that the output is not reversible.



$|\psi\rangle_g = \hat{U}_3\,\hat{U}_2\,\hat{U}_1\,|\psi\rangle_i$
→ Notice the order



Example of quantum circuit.

Why $\hat{U}_i$? U stays for UNITARY and each operator follows the property $U^\dagger = U^{-1}$ and $U^\dagger U = I$

$$U = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \xrightarrow{\dagger} U^\dagger = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}$$

$$|a|^2 + |b|^2 = 1 \qquad a^*c + b^*d = 0$$
$$|c|^2 + |d|^2 = 1 \qquad c^*a + d^*b = 0$$

$$U^\dagger U = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} |a|^2 + |b|^2 & a^*c + b^*d \\ c^*a + d^*b & |c|^2 + |d|^2 \end{pmatrix}$$

$$U|0\rangle = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \qquad \begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}$$

$$U|1\rangle \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

orthonormal vectors

Such system allows reversibility → I can apply the inverse of the U_i operator in order to go back.
HERMITIAN OPERATOR → $U = U^\dagger = U^{-1}$ $\quad U = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \longrightarrow U^\dagger = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}$ so $\begin{matrix} a = a^* \\ b = c^* \\ d = d^* \\ c = b^* \end{matrix}$

$U|\psi\rangle = \lambda|\psi\rangle$
$\langle\psi|U|\psi\rangle = \langle\psi|\lambda|\psi\rangle = \lambda$
$\langle\psi|U^\dagger = \langle\psi|\lambda^*$
$\langle\psi|U^\dagger|\psi\rangle = \lambda^*$ $\quad \lambda = \lambda^*$
$U = U^\dagger$

We introduced the Pauli operators → we now see the PAULI GATES:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ → X consists of a NOT port

$X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$ it actuates a BIT FLIP that on the block sphere consists of
$X|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$ a π rotation WRT X axis.

$$Y = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}$$ We see that Y operator does a BIT FLIP + phase SHIFT.

$Y|0\rangle = i|1\rangle$
$Y|1\rangle = -i|0\rangle$

$Z|0\rangle = |0\rangle$
$Z|1\rangle = -|1\rangle$ → Assume we don't know Z and I want to extract it:

$$Z = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

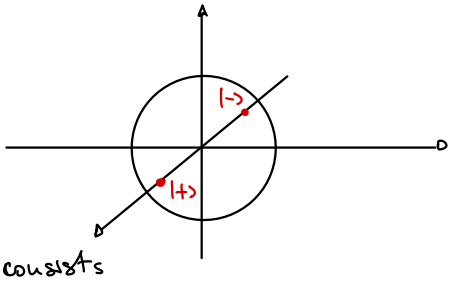
$$Z = |0\rangle\langle 0| - |1\rangle\langle 1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1\ 0) - \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0\ 1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In the end we have $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (identity)

HADAMARD OPERATOR

$\hat{H}|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$ PLUS STATE

$\hat{H}|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$ MINUS STATE



$\hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(\hat{x} + \hat{z})$ The Hadamard operator consists of a rotation WRT the bisector of y and z.

PHASE GATE

$\hat{R}_\theta|0\rangle = |0\rangle$ $\quad \hat{R}_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$ NOT HERMITIAN

$\hat{R}_\theta|1\rangle = e^{i\theta}|1\rangle$

For $\vartheta = \pi \rightarrow R_\theta = Z$

For $\vartheta = \frac{\pi}{2} \longrightarrow S = R_{\frac{\pi}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ → S is usually used in a "more symmetric" way

$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{pmatrix} = e^{i\frac{\pi}{4}}\begin{pmatrix} e^{-i\frac{\pi}{4}} & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}$

For $\vartheta = \frac{\pi}{4} \longrightarrow T = R_{\frac{\pi}{4}} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}}(1+i) \end{pmatrix}$

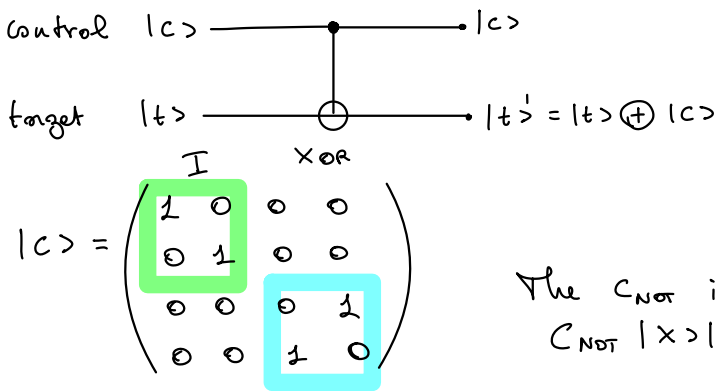
↳ For T instead $T = \begin{pmatrix} e^{-i\frac{\pi}{8}} & 0 \\ 0 & e^{+i\frac{\pi}{8}} \end{pmatrix}$

Generally a two qubit can be represented as follows:

$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle = \begin{pmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix}$ $\quad U = \begin{pmatrix} U_{00} & U_{01} & U_{02} & U_{03} \\ U_{10} & U_{11} & \cdots & \\ \vdots & & & \end{pmatrix}$

In this case the U operator must be a 4×4 matrix.

CONTROLLED NOT = CNOT



$|C\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

	$\|c\rangle$	$\|t\rangle$	$\|t\rangle'$
BUFFER	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
BUFFER	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
BIT FLIP	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
BIT FLIP	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$

↳ the X operator performs a NOT operation

The C_{NOT} is a sort of cloning operator

$C_{NOT}|x\rangle|0\rangle = |x\rangle|x\rangle$ CLONING works only for BASIS STATE

NON CLONING THEORY → in physics it is NOT possible to CLONE a state.

$\hat{U}_{copy}|\psi\rangle_1|0\rangle_2 = |\psi\rangle_1|\psi\rangle_2$

we have that the copy operator is copying the state of ψ in the one of 0. Is this possible?

$|\psi\rangle_1 = \alpha|0\rangle_1 + \beta|1\rangle_1$

$\hat{U}_{copy}(\alpha|0\rangle_1 + \beta|1\rangle_1)|0\rangle_2 = U_{copy}(\alpha|00\rangle + \beta|10\rangle) = \alpha|00\rangle + \beta|11\rangle$ → this is an ENTANGLED STATE

CONTRADDICTION

$|\psi\rangle_1|\psi\rangle_2 = (\alpha|0\rangle_1 + \beta|1\rangle_1)(\alpha|0\rangle_2 + \beta|1\rangle_2) =$ composite state (2 qubit state where I can factorize in 2 qubits)

↳ Here we don't have a superposition.

CONTROL - Z OPERATOR

$$C_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\|c\rangle$	$\|t\rangle$	$\|t'\rangle$
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|0\rangle$
$\|1\rangle$	$\|1\rangle$	$-\|1\rangle$

SWAP OPERATOR

$$SWAP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

SWAP

$|00\rangle \xrightarrow{\$} |00\rangle$

$|01\rangle \to |10\rangle$

$|10\rangle \to |01\rangle$

$|11\rangle \to |11\rangle$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} |01\rangle \\ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \end{matrix} = \begin{matrix} |10\rangle \\ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \end{matrix}$$

The SWAP allows to (RIVEDI)

TOFFOLI GATE (8 × 8)

$$|\psi\rangle = \begin{pmatrix} \alpha_{000} \\ \alpha_{001} \\ \\ \\ \\ \\ \\ \end{pmatrix}_{8\times 1}$$

We have a famous 3 bits GATE which is called TOFFOLI GATE (or CCNOT)

$$CC_{NOT} = \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix}$$

$|\psi\rangle$ — [measurement] → CLASSICAL STATE (BASIS)

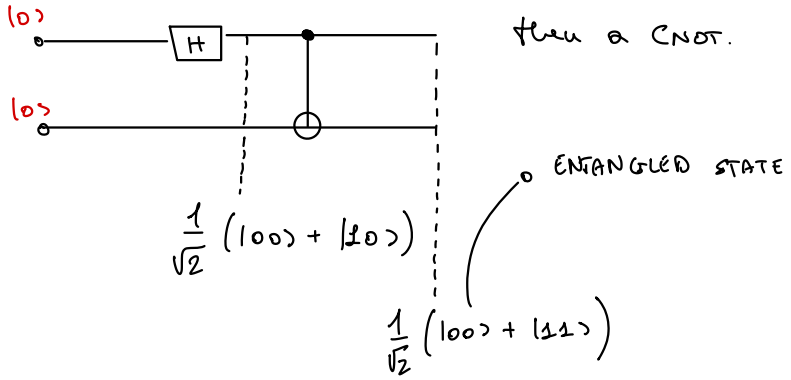
$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ — measurement → $|0\rangle\ (|\alpha|^2)$, $|1\rangle\ (|\beta|^2)$

Rivedi questa porta

BELL CIRCUIT

First Hadamard and then a CNOT.

Hadamard is very important, it first performs a superposition and then an entanglement



We now see a new topic. I start from the following state in order to understand what a rotation consists in.

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2} e^{i\theta}|1\rangle$$

$$\hat{R}_x = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

This is an ANTICLOCKWISE ROTATION BY θ ALONG X

I can have also $\hat{R}_y = \begin{pmatrix} \cos\frac{\vartheta}{2} & -\sin\frac{\vartheta}{2} \\ \sin\frac{\vartheta}{2} & \cos\frac{\vartheta}{2} \end{pmatrix}$ and $\hat{R}_z = \begin{pmatrix} e^{-i\frac{\vartheta}{2}} & 0 \\ 0 & e^{i\frac{\vartheta}{2}} \end{pmatrix}$ *

$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ → We mention the Pauli operators in order to show how these can be exploited in order to obtain other operators. In this case R_x, R_y and R_z

We obtain a general representation:

$\hat{R}_x = \hat{I}\cos\frac{\vartheta}{2} - i\,\text{sen}\frac{\vartheta}{2}\hat{\sigma}_x$

$\hat{R}_y = \hat{I}\cos\frac{\vartheta}{2} - i\,\text{sen}\frac{\vartheta}{2}\hat{\sigma}_y$

$\hat{R}_z = \begin{bmatrix} \cos\frac{\vartheta}{2} - i\,\text{sen}\frac{\vartheta}{2} & 0 \\ 0 & \cos\frac{\vartheta}{2} + i\,\text{sen}\frac{\vartheta}{2} \end{bmatrix}$ * $= \hat{I}\cos\frac{\vartheta}{2} - i\,\text{sen}\frac{\vartheta}{2}\hat{\sigma}_z$

$\hat{R}_u = \hat{I}\cos\frac{\vartheta}{2} - i\,\text{sen}\frac{\vartheta}{2}\hat{\sigma}_u =$

$= e^{-i\frac{\vartheta}{2}\hat{\sigma}_u}$

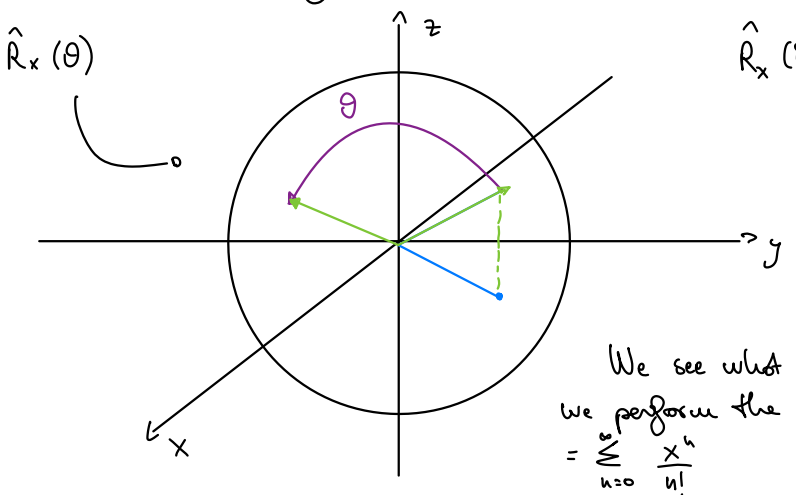
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

$e^{\hat{o}} = \hat{1} + \hat{o} + \frac{\hat{o}^2}{2!} + \frac{\hat{o}^3}{3!} + \ldots$

We will see that from this representation and by applying the series, we obtain the highlighted formula.

Most single qubit gates are dedicated to rotation.

Lecture 7 - 17/10/23



$$\hat{R}_x(\theta) = \hat{I}\cos\frac{\theta}{2} - i\,\hat{\sigma}_x \operatorname{sen}\frac{\theta}{2} = e^{-i\frac{\theta}{2}\hat{\sigma}_x}$$

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Euler's description

$$\cos\theta + i\operatorname{sen}\theta = e^{i\theta}$$

We see what it means to have an operator in the exponential:

we perform the Taylor expansion $\rightarrow e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Therefore we obtain: $e^{-i\frac{\theta}{2}\hat{\sigma}_x} = \sum_{n=0}^{\infty} \frac{\left(-i\frac{\theta}{2}\hat{\sigma}_x\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{\left(-i\frac{\theta}{2}\hat{\sigma}_x\right)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{\left(-i\frac{\theta}{2}\hat{\sigma}_x\right)^{2n+1}}{(2n+1)!} =$

$= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\theta}{2}\right)^{2n}}{(2n)!} \cdot \hat{I} - i \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\theta}{2}\right)^{2n+1}}{(2n+1)!} \hat{\sigma}_x \cdot \hat{I} =$ → We split in ODD and EVEN terms.

$\hat{\sigma}_x$ is a PAULI OPERATOR → if I apply it twice I obtain the Identity matrix. Therefore, $\hat{\sigma}_x$ elevated to an EVEN number gives me identity.

The first term is a SUMMATION of EVEN terms that compose the $\cos\frac{\theta}{2}$ while the second is the summation that composes $\operatorname{sen}\frac{\theta}{2}$.

There's a theory that states that in the worst case in which I would like to apply a quantum gate, I have to apply at most 3 rotations. Each gate can be represented by an evolution operator whose parameters are the angles of rotation along different axis.

$\begin{matrix} 1 & 2 & 3 \\ z & y & z \\ \lambda & \theta & \varphi \end{matrix}$ → $\hat{U}(\lambda,\theta,\varphi) = \hat{R}_z(\varphi)\hat{R}_y(\theta)\hat{R}_z(\lambda) = \begin{pmatrix} e^{-i\frac{\varphi}{2}} & 0 \\ 0 & e^{i\frac{\varphi}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} & -\operatorname{sen}\frac{\theta}{2} \\ \operatorname{sen}\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\lambda}{2}} & 0 \\ 0 & e^{i\frac{\lambda}{2}} \end{pmatrix} =$

↳ MOST GENERIC UNITARY operator

$= \begin{pmatrix} e^{-i\frac{\varphi}{2}} & 0 \\ 0 & e^{i\frac{\varphi}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2}e^{-i\frac{\lambda}{2}} & -\operatorname{sen}\frac{\theta}{2}e^{i\frac{\lambda}{2}} \\ \operatorname{sen}\frac{\theta}{2}e^{-i\frac{\lambda}{2}} & \cos\frac{\theta}{2}e^{i\frac{\lambda}{2}} \end{pmatrix} =$

$$\hat{U}_3 = \begin{pmatrix} \cos\frac{\theta}{2} & -\operatorname{sen}\frac{\theta}{2}e^{-i\lambda} \\ \operatorname{sen}\frac{\theta}{2}e^{i\varphi} & \cos\frac{\theta}{2}e^{i(\lambda+\varphi)} \end{pmatrix}$$

↳ one of the universal operators mostly used in quantum computing

$= \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\left(\frac{\lambda+\varphi}{2}\right)} & -\operatorname{sen}\frac{\theta}{2}e^{-i\left(\frac{\varphi-\lambda}{2}\right)} \\ \operatorname{sen}\frac{\theta}{2}e^{-i\left(\frac{\lambda-\varphi}{2}\right)} & \cos\frac{\theta}{2}e^{i\left(\frac{\lambda+\varphi}{2}\right)} \end{pmatrix}$

How do we obtain other operators from this one:

θ	φ	λ	U_3
0	0	0	$\hat{I}$
π	0	π	$\hat{\sigma}_x$
π	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\hat{\sigma}_y$
0	0	π	$\hat{\sigma}_z$
$\frac{\pi}{2}$	0	π	$\hat{H}$

For a rotation along z we just need 1 rotation

Evolution operator describes how my wave is evolving over time.

$\hat{H}|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle$ $\quad$ $|\psi(t)\rangle = e^{-i\frac{\hat{H}}{\hbar}t}|\psi(0)\rangle = \hat{U}|\psi(0)\rangle$

$\frac{d}{dt}|\psi\rangle = -i\frac{\hat{H}}{\hbar}|\psi\rangle$ $\quad$ $\frac{df}{dt} = -\alpha f$ $\quad$ $f(t) = e^{-\alpha t}f(0)$ $\quad$ $|\psi\rangle_f = \hat{U}|\psi\rangle_i$

$e^{-i\frac{\hat{H}}{\hbar}t} = e^{-i\frac{\theta}{2}\hat{\sigma}_z} = e^{-i\frac{\omega_0 t}{2}\hat{\sigma}_z}$

↳ I want to impose such relation → $\theta = \omega_0 t$

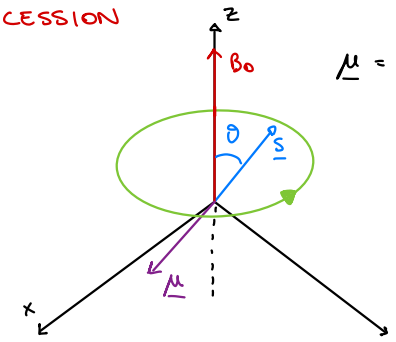
$\hat{H} = \frac{\hbar\omega_0}{2}\hat{\sigma}_z$ My Hamiltonian should have such expression. I don't know actually how to do it.

SPIN COMPONENT

SPIN OPERATOR

$\hat{H} = \omega_0 \hat{S}_z$

PRECESSION



$\underline{\mu} = \gamma\,\underline{S} = g\cdot\frac{q}{2m}\cdot\underline{S}$

↳ around 2 for the electron

q is negative → μ is opposite WRT $\underline{S}$

$\underline{N} = \underline{\mu} \times \underline{B_0} = \gamma\,\underline{S} \times \underline{B_0} = -\gamma\,\underline{B_0} \times \underline{S}$

↳ TORQUE OF μ

↓

I extract the minus and flip the vectorial product

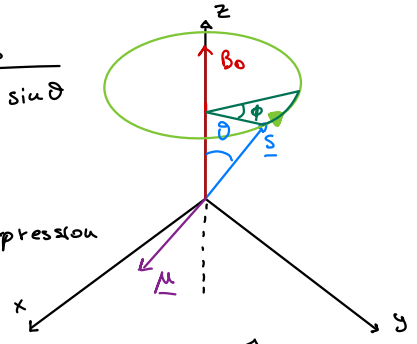
In green we have the anti clock-wise rotation around z due to the magnetic field.

$\frac{dS}{dt} = N$ $\quad B_0 \cdot \frac{dS}{dt} = 0$ → The torque is always perpendicular to the field

$B_0 \cdot \frac{dS}{dt} = \frac{d}{dt}(B_0 \cdot S)$ The derivative of the angle between S and B_0 is zero → it remains CONSTANT → This aspect is called PRECESSI

LARMOR PRECESSION

We can define the Larmor Frequency $\omega_0 = \frac{d\phi}{dt} = \frac{dS}{dt\, S \sin\theta}$



We know that $\hat{H} = -\mu \cdot B_0 = -\gamma S \cdot B_0 = \quad \omega_0 = -\gamma B_0$

$= -\gamma B_0 \hat{S}_z = \omega_0 \hat{S}_z$

I use S_z as an operator since I need the expression for the Hamiltonian $\hat{H}$ as operator.

Let's see which are the eigenvalues and eigenvectors of my operator $\hat{H}$

$\hat{H}|\psi\rangle = \varepsilon|\psi\rangle \qquad \frac{\hbar\omega_0}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar\omega_0}{2}\begin{pmatrix}1\\0\end{pmatrix}$

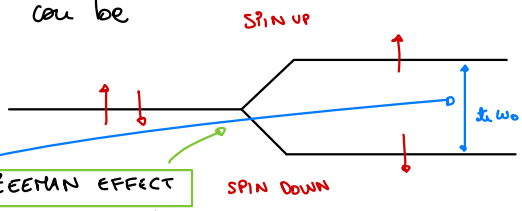
↳ Energy of the $|0\rangle$ basys state

$\hat{H}|0\rangle = E_0|0\rangle$

$\frac{\hbar\omega_0}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = -\frac{\hbar\omega_0}{2}\begin{pmatrix}0\\1\end{pmatrix}$

↳ $\hat{H}|1\rangle = \varepsilon_1|1\rangle$

If I have an electron state it can be



↳ Based on the splitting we can perform a spin to charge conversion

What I'm really interested about is the superposition state of the qubit.

$\hat{H}|\psi\rangle = i\hbar\frac{d}{dt}|\psi\rangle$

$|\psi\rangle = c_0|0\rangle e^{-i\frac{E_0}{\hbar}t} + c_1|1\rangle e^{-i\frac{\varepsilon_1}{\hbar}t} = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2} e^{i\frac{E_0-E_1}{\hbar}t}|1\rangle$

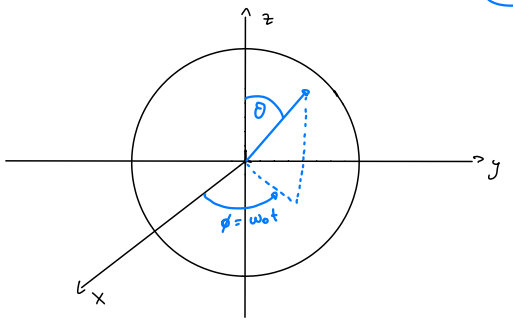
$|c_0|^2 + |c_1|^2 = 1$

$c_0 = \cos\frac{\theta}{2}$

$c_1 = \sin\frac{\theta}{2}$

$= \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2} e^{i\omega_0 t}|1\rangle$

PRECESSION The angle WRT z remains constant but I have a rotation along it of the state.



$\Delta E = \hbar\omega_0 = -\hbar\gamma B_0 = -2\left[\frac{\hbar}{2} g \frac{q}{2m}\right] B_0 = 2\mu_B B_0$

BOHR MAGNETON (universal constant $\Rightarrow \mu_B = 9{,}274 \cdot 10^{-24}\ \frac{J}{T}$)

Let's assume $B_0 = 1T$

$\Delta E = 2 \cdot 10^{-23} J = \frac{2 \cdot 10^{-23}}{1{,}6 \cdot 10^{-19}} eV = 120\,\mu eV$

$KT \ll \Delta E$ → Otherwise the thermal energy would destroy my state and coherence

$T \ll \frac{\Delta E}{K} = 1{,}34\,K$

$\omega_0 = \frac{\Delta E}{\hbar} \simeq \frac{2 \cdot 10^{-23}}{10^{-34}} \frac{J}{Js} = 200\,G\,\frac{rad}{s}$

$T = \frac{2\pi}{200}\,ns = 30ps$

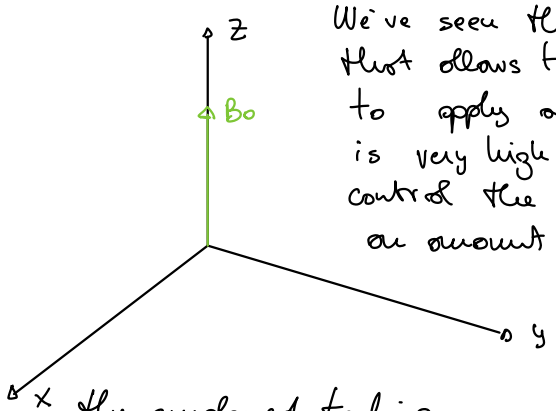
TOO FAST PERIOD FOR SWITCHING TIME MAGNETIC FIELD OF 1T

we are using $\hbar$ instead of h, therefore we have 2π factor

A quantum gate is represented by a unitary operator $\hat{U}$ and always corresponds to a rotation in the block sphere.

Lecture 8 - 19/10/23

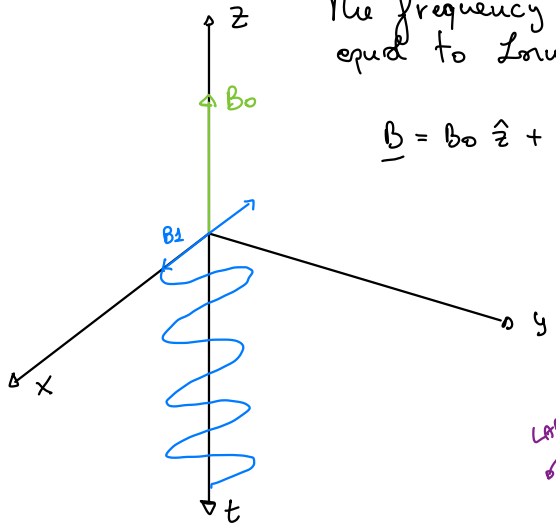
Let's assume to have a static magnetic field along z-axis -> we've seen that such field causes a precession of any qubit state (in this case we're dealing with a SPIN qubit but it can be extended to any kind of qubit.



We've seen that in order to obtain a reasonable Zeeman SPLITTING that allows to recognize the positive or negative spin, we need to apply a very large magnetic field and the precession speed is very high -> $\omega_0 = -\gamma B_0 \approx$ GHz -> we have that in order to control the rotation we should turn ON a very large B_0 for an amount of time equal to few ps! (Very difficult to implement)

We need to think about something else: ELECTRON SPIN RESONANCE. In general we call the employed technique as DRIVEN RABI OSCILLATIONS (which can be applied to any kind of qubit).

What is electron spin resonance? Let's assume to apply an oscillating magnetic field that moves over time along x (apart from the already applied B_0).



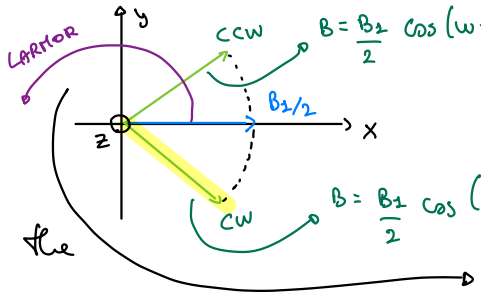
The frequency with which B_1 switches from + to - is almost equal to Larmor precession frequency. The total field is then:

$$\underline{B} = B_0 \hat{z} + B_1 \cos(\omega t + \delta) \hat{x}$$

↳ PHASE

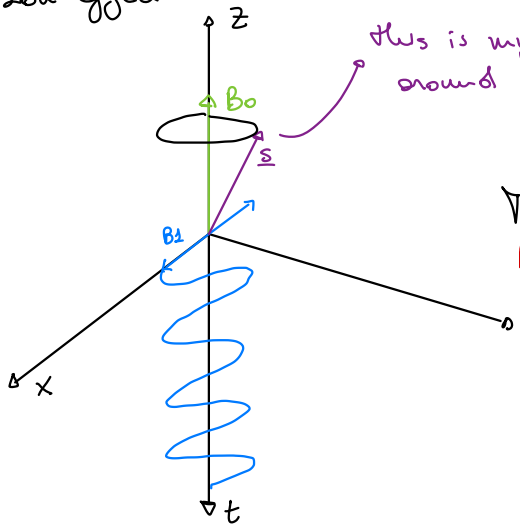
↳ we are actually considering a general frequency

By looking at xy plane, we have that B_1 wave can also be described as sum of components rotating in clockwise and anticlockwise directions WRT y.



As we see one component follows the same clockwise direction of the LARMOR precession. The other component is instead moving at twice the frequency ω_0 (WRT the Larmor, since we do a summation of vectors speed, it results, by assuming ω_0 also for B_1, twice the speed).

Why did we decompose the B_1 vector? Because we want to see the precession effect



This is my qubit state that is rotating around z due to precession

Therefore we introduce the ROTATING WAVE APPROXIMATION (RWA)

↓

Since the fast wave is moving so fast WRT the system I'm observing, it averages to zero.

↓

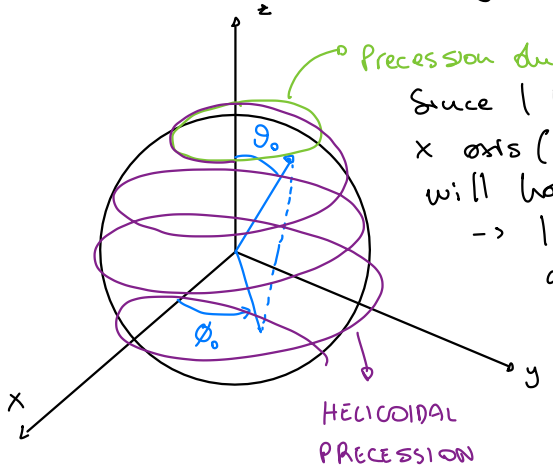
The clockwise component gets neglected

We have that two precessions take place → one referred to the B_0 and then other to B_1, more exactly to B_1 that, even if it is rotating, it results constant WRT our reference system. (Note that I'm assuming $w = w_0$)
Moreover note that $B_1 \ll B_0$ $B_1 \simeq 10^{-3} B_0$.

RABI FREQUENCY

The precession along x takes place with a frequency $\Omega = -\gamma \frac{B_1}{2}$ MHrad/s → Here I can I can tune the B_1 field in order to obtain the wanted rotation → do controllable rotation since I don't operate → it is easier to do since $B_1 \ll B_0$ and at the same time we have to work with reasonable frequencies. anymore in ps range but in ns range. (much more reasonable)

Let's see what's happening in the block sphere:



Precession due to B_0

Since I have a second precession along the x axis (much slower than B_0 precession) I will have a sort of helicoidal precession → → I see that in this case I'm able to control also ϑ_0 instead of just ϕ_0!

I can write my superposition state as:

$$|\psi\rangle = \cos\frac{\vartheta_0 + \Omega t}{2}|0\rangle + \sin\frac{\vartheta_0 + \Omega t}{2} e^{i(\phi_0 + w_0 t)}|1\rangle$$

We've a rotation ← $\hat{x}$ ($\hat{y}$) along $\hat{x}$ that we are able to tune. We have some dependence from $\hat{y}$ rotation. We'll see how to properly control the two.

RABI

I have to add also precession LARMOR FREQUENCY

We need now to write the Hamiltonian for this new kind of field. In this case we have a problem of TIME DEPENDENCE.

In the previous lesson we saw that: $U = -\underbrace{\gamma \underline{S}}_{\text{this is essentially the magnetic dipole}} \cdot \underline{B}$ → My Hamiltonian could be written as: $\hat{H} = -\gamma \frac{\hbar}{2} \hat{\sigma}_z B_0$ where this element is the Bohr magneton → $\hat{H} = \mu_B B_0 \hat{\sigma}_z$

In the new situation my field is depending on time, therefore I define $\hat{H}'$
Before giving a precise definition of $\hat{H}'$, let's remind:

$$|0\rangle \longrightarrow +\mu_B B_0 = \frac{\hbar w_0}{2}$$
$$|1\rangle \longrightarrow -\mu_B B_0 = -\frac{\hbar w_0}{2}$$

From this I can write the superposition state:
$\hat{H}|\psi\rangle = i\hbar \frac{\partial}{\partial t}|\psi\rangle$

$$|\psi\rangle = \begin{pmatrix} \alpha e^{-i\frac{E_0}{\hbar}t} \\ \beta e^{-i\frac{E_1}{\hbar}t} \end{pmatrix} *$$

α can be seen as $\cos\frac{\vartheta}{2}$ and β as $\sin\frac{\vartheta}{2}$

If we have a time dependent field I cannot rely on the approach we employed until now: we cannot separate a time dependent Schrödinger equation into a non dependent + equation only in time.

this state is doing precession → we can take the phase factor out and notice:

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta e^{i w_0 t} \end{pmatrix}$$ (we've taken the global phase out and we see a time dependent rotating factor).

Now I have B_1 moving in time → I need to rely on PERTURBATION THEORY: it says that I have a perturbation Hamiltonian that is applied on top of the standard one

$$\hat{H}' = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$$

The new equation will look like $(\hat{H} + \hat{H}')|\psi\rangle = i\hbar \frac{\partial}{\partial t}|\psi\rangle$ but, the perturbation theory tells that in case of WEAK PERTURBATION (that is our case since B_1 is very small) then $\hat{H}' \ll \hat{H}$ and we can rely on the expression *

found before, provided that α and β are now dependent on time!

$$|\psi\rangle = \begin{pmatrix} \alpha(t)\, e^{-i\frac{E_0}{\hbar}t} \\ \beta(t)\, e^{-i\frac{E_1}{\hbar}t} \end{pmatrix}$$

Going back to U we now have $U = -\gamma \underline{S}\cdot\underline{B} = \mu_B \cdot B_1 \cos(\omega t+\delta)\,\hat{\sigma}_x$

$$\underbrace{\hat{H}+\hat{H}'}_{\text{complete Hamiltonian}} = \mu_B \begin{pmatrix} B_0 & B_1\cos(\omega t+\delta) \\ B_1(\cos\omega t+\delta) & -B_0 \end{pmatrix}$$

I need now to solve the equation:

$$(\hat{H}+\hat{H}')|\psi\rangle = i\hbar \frac{d}{dt}|\psi\rangle$$

DIFFERENTIAL EQUATION SOLUTION:

1) $\mu_B B_0 \alpha\, e^{-i\frac{E_0}{\hbar}t} + \mu_B B_1 \cos(\omega t+\delta)\,\beta\, e^{-i\frac{E_1}{\hbar}t} = i\hbar \frac{d\alpha}{dt} e^{-i\frac{E_0}{\hbar}t} + \alpha E_0 e^{-i\frac{E_0}{\hbar}t}$

(remember it is $\alpha(t)$)

$(-i\cdot i = +1)$

$\hbar$ gets simplified

note that this should be a partial derivative.

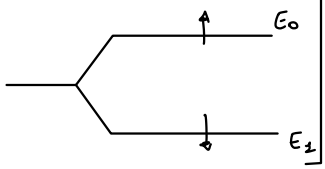
RECALL:

The precession Hamiltonian used to be:

$$\mu_B \begin{pmatrix} B_0 & 0 \\ 0 & -B_0 \end{pmatrix} |\psi\rangle = E|\psi\rangle$$

$|0\rangle$ $E_0 = \mu_B B_0$ } ZEEMANN SPLITTING

$|1\rangle$ $E_1 = -\mu_B B_0$ }



We realize that $E_0 = \mu_B B_0$ and therefore from equation 1 we can simplify two terms

2) $\mu_B B_1 \alpha \cos(\omega t+\delta)\, e^{-i\frac{E_0}{\hbar}t} - \mu_B B_0 \beta e^{-i\frac{E_1}{\hbar}t} = i\hbar \frac{d\beta}{dt} e^{-i\frac{E_1}{\hbar}t} + \beta E_1 e^{-i\frac{E_1}{\hbar}t}$

$\hookrightarrow E_1 = -\mu_B B_0$ (as before we cancel the two terms)

FROM 1)

$$i\hbar\frac{d\alpha}{dt} = \mu_B B_1 \beta \cos(\omega t+\delta) e^{i\omega_0 t} = \frac{\mu_B B_1}{2}\beta\left(e^{i(\omega+\omega_0)t} e^{i\delta} + e^{-i(\omega-\omega_0)t} e^{-i\delta}\right)$$

$\frac{1}{2}$ of the cosine; $e^{-i(\omega-\omega_0)t}$: 1

FROM 2)

$$i\hbar\frac{d\beta}{dt} = \mu_B B_1 \alpha \cos(\omega t+\delta) e^{-i\omega_0 t} = \frac{\mu_B B_1}{2}\alpha\left(e^{i(\omega-\omega_0)t} e^{i\delta} + e^{-i(\omega+\omega_0)t} e^{-i\delta}\right)$$

$e^{i(\omega-\omega_0)t}$: 1

WHY DO WE CANCEL THE TERMS? Because as we've seen before if we have a wave moving at a frequency 2ω WRT ω of observation, its AVG = 0

We do a first approximation → we assume $\omega = \omega_0$ (we work at resonance) in red the consequence

Remember that we introduced the RABI frequency

FROM 1)

$$i\hbar\frac{d\alpha}{dt} = \frac{\mu_B B_1}{2}\cdot e^{-i\delta}\beta \longrightarrow \frac{d\alpha}{dt} = -i\frac{\mu_B B_1}{2\hbar} e^{-i\delta}\beta$$

FROM 2)

$$i\hbar\frac{d\beta}{dt} = \frac{\mu_B B_1}{2}\cdot e^{i\delta}\alpha \longrightarrow \frac{d\beta}{dt} = -i\frac{\mu_B B_1}{2\hbar} e^{i\delta}\alpha$$

By substitution I obtain:

$$\frac{d^2\alpha}{dt} = -i\frac{\mu_B B_1}{2\hbar} e^{-i\delta}\frac{d\beta}{dt} = -\left(\frac{\mu_B B_1}{2\hbar}\right)^2 \alpha = -\left(\frac{\Omega}{2}\right)^2\alpha$$

As we see the second derivative of α is still α but with inverted sign

$\hookrightarrow \frac{\Omega}{2}$ → Rabi frequency $= \frac{\mu_B B_1}{\hbar}$

we can write α in the form of a cosine.

$$\alpha(t) = \cos\left(\frac{\Omega}{2}t\right)$$

$$|\psi\rangle = \begin{pmatrix} \cos\left(\frac{\Omega}{2}t\right) e^{-i\frac{E_0}{\hbar}t} \\ -i\,\mathrm{sen}\left(\frac{\Omega}{2}t\right) e^{i\delta} e^{-i\frac{E_1}{\hbar}t} \end{pmatrix}$$

For $t=0$ $|\psi\rangle = |0\rangle$

FINAL RESULT

For what concerns β:

$$\beta = -i\frac{\Omega}{2}\,\mathrm{sen}\left(\frac{\Omega}{2}t\right)e^{i\delta} = -i\,\mathrm{sen}\left(\frac{\Omega}{2}t\right)e^{i\delta}$$

$\frac{\Omega}{2}$

I can assume $\delta = 0$ (NO PHASE) and rewrite $|\psi\rangle$ as:

$$|\psi\rangle|_{\delta=0} = \begin{pmatrix} \cos\frac{\Omega}{2}t \\ -i\,\mathrm{sen}\frac{\Omega}{2}t\, e^{i\omega_0 t} \end{pmatrix}$$

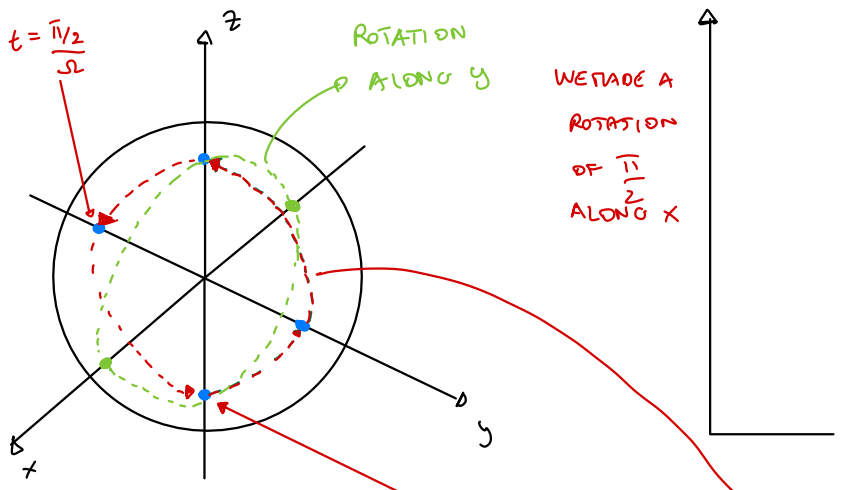
I realize that I have a precession along z and also along x.

LARMOR PRECESSION at ω_0

RABI PRECESSION at $\frac{\Omega}{2}$

What happens in terms of rotation?

Assume that $t = \frac{\pi/2}{\Omega} \longrightarrow |\psi\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -i\,1/\sqrt{2} \end{pmatrix}$ → it is a superposition state lying on x, y plane



Remember that:

$$|+\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$|-\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$|+i\rangle = \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}$$

$$|-i\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}$$

If we assume instead $t = \frac{\pi}{\Omega} \longrightarrow |\psi\rangle = \begin{pmatrix} 0 \\ -i \end{pmatrix}$

By continuing to increase t we obtain a complete rotation around x.

Note that we assume $\delta = 0$ → if we complicate the things and assume for example $\delta = \frac{\pi}{2}$ → in this case we see that the rotation is no more around x but around y.

$$|\psi\rangle = \begin{pmatrix} \cos\frac{\Omega}{2}t \\ \operatorname{sen}\frac{\Omega}{2}t\, e^{-i\left(\frac{\pi}{2} - \frac{\pi}{2}\right)} \end{pmatrix} = \begin{pmatrix} \cos\frac{\Omega}{2}t \\ \operatorname{sen}\frac{\Omega}{2}t \end{pmatrix} \longrightarrow \text{after } t = \frac{\pi}{2}/\Omega \longrightarrow |\psi\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

See the green rotation

As we see, by playing with the phase we are able to implement a universal gate able to perform the rotations needed.

$\Omega t = \theta$

↳ We can play with Ω and t in order to choose the θ of rotation

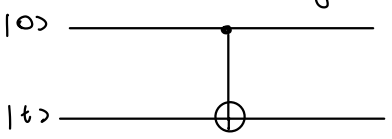
We have up until now seen how to deal with 1 qubit gates. Now we approach the 2 qubits approach. Lecture 9 - 31/10/23

1 QBIT GATE → ELECTRON SPIN RESONANCE (applies to SPIN qubit but also to any kind of qubit)

We've seen that it is possible to perform arbitrary rotations along x and y, while along z we have precession. By changing phase we have the possibility to change the rotation axis.

For any quantum computer we need a UNIVERSAL SET OF QUANTUM GATES. I have to establish a minimum amount of quantum gates I can use in order to implement any functions/algorithms

The CNOT is a kind of 2QBIT GATE but we don't consider it as 2 QBIT GATE.



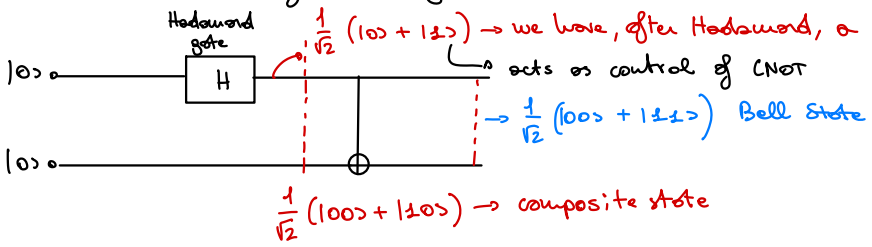
If control is $|0\rangle$ ($|c\rangle = |0\rangle$) ⟹ $|t\rangle$ UNCHANGED

" $|1\rangle$ ($|c\rangle = |1\rangle$) ⟹ $|t\rangle$ FLIPPED

$$\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The 2 QBIT gates are important since they lead to ENTANGLEMENT → quantum computing is important for 2 characteristics that are SUPERPOSITION (any qubit can be either $|0\rangle$ or $|1\rangle$ or a superposition $\alpha|0\rangle + \beta|1\rangle$) and ENTANGLEMENT (the state of 1 qubit is associated to the state of a second qu-bit). A typical example of ENTANGLEMENT is the BELL STATE $|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

Such state is created by means of a BELL CIRCUIT:



SWAP → First and second qubit are exchanged

$|00\rangle \longrightarrow |00\rangle$

$|01\rangle \longrightarrow |10\rangle$

$|10\rangle \longrightarrow |01\rangle$

$|11\rangle \longrightarrow |11\rangle$

$$\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

example

iSWAP (in addition to swap we introduce a phase shift)

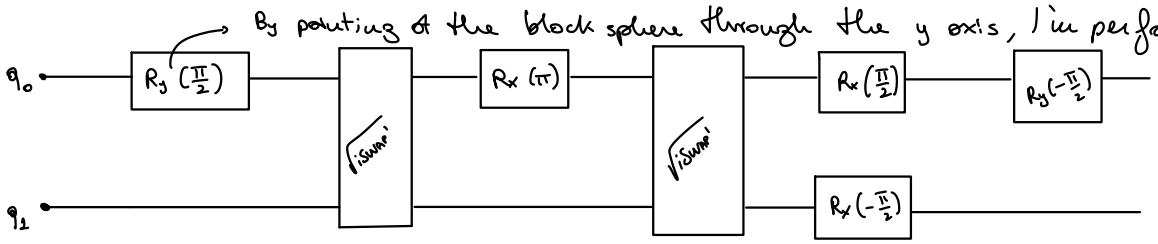
$$\text{iSWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \sqrt{\text{iSWAP}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \sqrt{\text{iSWAP}} \cdot \sqrt{\text{iSWAP}} = \text{iSWAP}$$

$$\sqrt{\text{iSWAP}'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

→ By doing $\sqrt{\text{iSWAP}'} \cdot \sqrt{\text{iSWAP}'} = \text{iSWAP}'$ → Conjugate of iSWAP

We now want to explore the physical events that make us able to realize such iSWAP' gate.

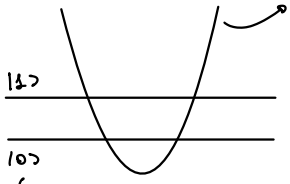
We'll see that from iSWAP' I can obtain CNOT.



By pointing at the block sphere through the y axis, I'm performing an anticlockwise rotation of $\frac{\pi}{2}$.

Such entire circuit is the one implementing the CNOT gate.

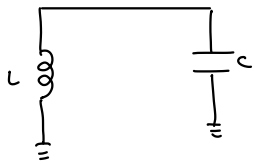
We assume to have two basys states $|0\rangle$ and $|1\rangle$ at two different potentials (according to wave function we must have a potential.



we've a sort of contract potential, like an harmonic oscillator (RESONATOR in quantum computing).

↳ In quantum computing doesn't have resistances (since we work with superconducting materials)

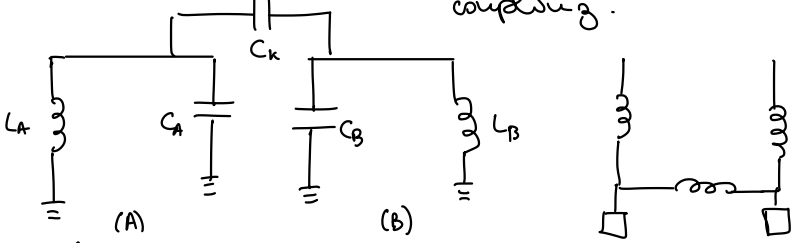
↳ At low temperature it doesn't work as an oscillator but as a quantum resonator (almost loss less)



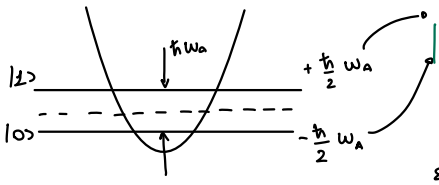
↳ When we talked about Zeeman splitting we had that $|1\rangle$ was the state pointing to a low level of potential while $|0\rangle$ to the higher level.

↳ It is possible to show that such phenomena is equal to the effect of a quantum gate where I implement an 'iSwap'
We then describe such couple qubit case by means of oscillators coupling.

If I have such situation with a capacitive coupling.



They represent two qubits. If I couple them capacitively, I obtained a coupled oscillator.
It is like connecting two "pendulum" (mechanical equivalent of the resonator). The capacitive connection is like connecting the two pendulums with a spring.
Such coupling gives rise to an interesting event: the energy can be transfered from one resonator to the other



(A)

Eigen values (energies) associated to the basys states ω_A is the characteristic frequency of an resonator.

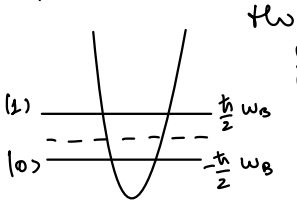
I can describe this resonator with an Hamiltonian:

$$-\frac{\hbar}{2}\omega_A \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_z} |\psi_A\rangle = i\hbar \frac{\partial}{\partial t} |\psi_A\rangle$$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Until now we've analyzed the situation for the single qubit, let's step to a second qubit:
this second qubit is, for now, uncoupled from the previous and shows a different potential with different ω characteristic



(B)

So, our qubits are not interacting one each other and there's no coupling. I can write:

$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ ← Tensor product

My 2-qubit Hamiltonian can be instead written in this way:

$\hat{H} = \hat{H}_A \otimes \hat{I} + \hat{I} \otimes \hat{H}_B$

↳ $\hat{H}_A = -\frac{\hbar}{2}\begin{pmatrix} \omega_A & 0 \\ 0 & -\omega_A \end{pmatrix}$ (from the wave function equation written before)

$\hat{H}_B = -\frac{\hbar}{2}\begin{pmatrix} \omega_B & 0 \\ 0 & -\omega_B \end{pmatrix}$

We want to obtain the Hamiltonian for 2-qubits:

$$\hat{H}_A \otimes \hat{I} = -\frac{\hbar}{2}\begin{pmatrix} \omega_A & 0 & 0 & 0 \\ 0 & \omega_A & 0 & 0 \\ 0 & 0 & -\omega_A & 0 \\ 0 & 0 & 0 & -\omega_A \end{pmatrix}$$

means that each element gets multiplied for each element of the $\hat{I}$ matrix.

$$\hat{I} \otimes \hat{H}_B = -\frac{\hbar}{2}\begin{pmatrix} \omega_B & 0 & 0 & 0 \\ 0 & -\omega_B & 0 & 0 \\ 0 & 0 & \omega_B & 0 \\ 0 & 0 & 0 & -\omega_B \end{pmatrix}$$

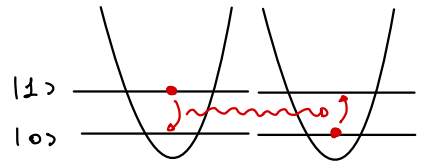
$$\hat{H} = \hat{H}_A \otimes \hat{I} + \hat{I} \otimes \hat{H}_B = -\frac{\hbar}{2}\begin{pmatrix} \omega_A+\omega_B & 0 & 0 & 0 \\ 0 & \omega_A-\omega_B & 0 & 0 \\ 0 & 0 & -\omega_A+\omega_B & 0 \\ 0 & 0 & 0 & -\omega_A-\omega_B \end{pmatrix}$$

eigenvalue of 00 state — eigenvalue of 01 state — 10 — 11

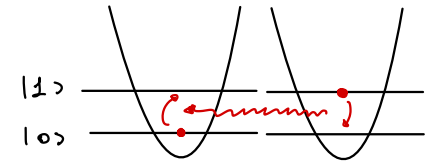
Now I need some kind of perturbation I need, more precisely to perform two operations: the first one is to couple the oscillations and get a fine tuning of the oscillations → If I'm not able to tune them I don't obtain the coupling.

Why do I need the frequencies ω_A and ω_B to be very similar in order to get a proper coupling? Because I won't have energy transfer. This aspect is fundamental in order to obtain a 2-qubits. I must have an energy transfer that allows to reach a balance between the two energies.

Let's assume to have a situation like the one shown here on the left where the first qubit can transfer its energy to the second state by means of an electromagnetic wave. Such transfer consists in a SWAP $|10\rangle \longrightarrow |01\rangle$ and it is possible only in case of a correct energy transfer → thus $\omega_B \simeq \omega_B$



CASE $|10\rangle \xrightarrow{\text{SWAP}} |01\rangle$



CASE $|01\rangle \xrightarrow{\text{SWAP}} |10\rangle$

So now we couple the two qubits: we add k, a coupling factor, inside the matrix

$$\hat{H} = \hat{H}_A \otimes \hat{I} + \hat{I} \otimes \hat{H}_B = -\frac{\hbar}{2}\begin{pmatrix} \omega_A+\omega_B & 0 & 0 & 0 \\ 0 & \omega_A-\omega_B & k & 0 \\ 0 & k & -\omega_A+\omega_B & 0 \\ 0 & 0 & 0 & -\omega_A-\omega_B \end{pmatrix}$$

This couples the 10 to 01

Such NON-DIAGONAL k terms is used to represent the interaction between the 01 and 10 states. I can rewrite my 2 qubit Hamiltonian, focusing on $|01\rangle$ and $|10\rangle$ states.

How can I write my eigenfunctions? If I want to find the eigenfunction for $|\psi_{01}\rangle$ and $|\psi_{10}\rangle$, I will have that it won't be a solution of Schrödinger's equation, unless we make some assumptions. The assumption we make is the SMALL PERTURBATION one that allows we to write the eigenfunction as a superposition one: $|\psi\rangle = C_{01}|\psi_{01}\rangle + C_{10}|\psi_{10}\rangle$ where C_{01} and C_{10} are constant values accounting for the applied perturbation. UNCOUPLED EIGENFUNCTION → This is allowed from the SMALL PERTURBATION THEORY and it states that we can assume $|\psi_{01}\rangle$ and $|\psi_{10}\rangle$ will remain almost unchanged during time.

From this we can write a simpler COUPLED EIGENFUNCTION: $|\psi\rangle = C_{01}(t)|\psi_{01}\rangle + C_{10}(t)|\psi_{10}\rangle$ where the perturbation isn't anymore constant.

With these assumptions we can see how the two terms composing $|\psi\rangle$, satisfy Shrödinger's equation

$\Delta = \omega_A - \omega_B \to 0$ my eigenfunction can be written as a state $|01\rangle$ energy traveling as a wave.

$|\psi_{01}\rangle = |01\rangle \cdot e^{i\frac{\omega_A-\omega_B}{2}t}$

We have then that $|01\rangle \cdot i\hbar \frac{\partial C_{01}}{\partial t}|\psi_{01}\rangle - \hbar C_{01}\frac{\omega_A-\omega_B}{2}|\psi_{01}\rangle = -\frac{\hbar}{2}(\omega_A-\omega_B)|\psi_{01}\rangle + k|01\rangle\langle 10|C_{10}|\psi_{10}\rangle$

I obtain a differential equation where it is possible to see the coupling. → I can write my k as a CAT and a BRA

The second equation is almost the same:

$|\psi_{10}\rangle = |10\rangle \cdot e^{-i\frac{\omega_A-\omega_B}{2}t}$

$|10\rangle : i\hbar\frac{\partial C_{10}}{\partial t}|\psi_{10}\rangle + \hbar C_{10}\frac{\omega_A-\omega_B}{2}|\psi_{10}\rangle = k|10\rangle\langle 01|C_{01}|\psi_{01}\rangle + \frac{\hbar}{2}(\omega_A-\omega_B)C_{10}|\psi_{10}\rangle$

$$k\begin{pmatrix}0\\1\\0\\0\end{pmatrix}(00\ 10) = k\begin{pmatrix}0&0&0&0\\0&0&k&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

So we have:

$$\begin{cases} i\hbar\frac{\partial C_{01}}{\partial t}e^{i\frac{\omega_A-\omega_B}{2}t}|01\rangle = kC_{10}e^{-i\frac{\omega_A-\omega_B}{2}t}|01\rangle \\ i\hbar\frac{\partial C_{10}}{\partial t}e^{-i\frac{\omega_A-\omega_B}{2}t}|10\rangle = kC_{01}e^{i\frac{\omega_A-\omega_B}{2}t}|10\rangle \end{cases}$$

Remember that $|\psi_{10}\rangle = |10\rangle e^{-i\frac{\omega_A-\omega_B}{2}t}$ and $|10\rangle\langle 10| = 1$

$|\psi_{01}\rangle = |01\rangle e^{i\frac{\omega_A-\omega_B}{2}t}$ $|01\rangle\langle 01| = 1$

NEXT PAGE

$$i\hbar \frac{\partial C_{01}}{\partial t} = K C_{10} e^{i\Delta t} \quad (\text{where } \Delta = \omega_A - \omega_B)$$

$$i\hbar \frac{\partial C_{10}}{\partial t} = K C_{01} e^{-i\Delta t}$$

→ This is the simplest representation of my differential equation. I can solve it by applying a second derivative

$$-\hbar^2 \frac{\partial^2 C_{01}}{\partial t^2} = i\hbar K \frac{\partial C_{10}}{\partial t} e^{i\Delta t} - \Delta\hbar K C_{10} e^{i\Delta t} = [K^2 C_{01}] - \left[i\hbar\Delta \frac{\partial C_{01}}{\partial t}\right]$$

→ can be written in terms of C_{01}

$$-\hbar^2 \frac{\partial^2 C_{10}}{\partial t^2} = i\hbar K \frac{\partial C_{01}}{\partial t} e^{-i\Delta t} + \Delta\hbar K C_{01} e^{-i\Delta t} = \text{almost similar}$$

$$i\hbar \frac{\partial C_{10}}{\partial t} = K C_{01} e^{-i\Delta t}$$

$$K e^{i\Delta t} \cdot \frac{\partial}{\partial t} K C_{01} \cdot e^{-i\Delta t} =$$

$$= K^2 C_{01}$$

$$K C_{10} \cdot e^{i\Delta t} = i\hbar \frac{\partial C_{01}}{\partial t}$$

$$\Delta\hbar K C_{10} \cdot e^{i\Delta t} = \Delta\hbar \cdot$$

$$\cdot i\hbar \frac{\partial C_{01}}{\partial t} = i\hbar^2 \Delta \frac{\partial C_{01}}{\partial t}$$

We have obtained a differential equation of 2nd order only in function of C_{01} (and C_{10})

$$\frac{\partial^2 C_{01}}{\partial t^2} - i\Delta \frac{\partial C_{01}}{\partial t} + \left(\frac{K}{\hbar}\right)^2 C_{01} = 0$$

Then we remember that we are operating in a tuning condition for which $\Delta \simeq 0$ since $\omega_A - \omega_B = \Delta$ and $\omega_A \simeq \omega_B$. (otherwise no transfer)

$\frac{K}{\hbar} = \Omega$ (angular frequency)

$C_{01}(t) = A \cos\Omega t + B \sin\Omega t$

$C_{01}(0) = A$

$$\left.\frac{\partial C_{01}}{\partial t}\right|_0 = B\Omega = -i\frac{K}{\hbar} C_{10}(0) = -i\Omega C_{10}(0) \qquad B = -i C_{10}(0)$$

(Ω over $\frac{K}{\hbar}$)

↑ I can rewrite the previous derivative in terms of $B\Omega$

Therefore:

$$C_{01}(t) = C_{01}(0)\cos\Omega t - i C_{10}(0) \operatorname{sen}\Omega t$$

$$C_{10}(t) = -i C_{01}(0) \operatorname{sen}\Omega t + C_{10}(0)\cos\Omega t$$

↓

We can write at this point the MATRIX for the 2-qubit unitary operator:

$$\hat{U} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Omega t & -i\operatorname{sen}\Omega t & 0 \\ 0 & -i\operatorname{sen}\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

→ At the boundaries we have the situation for the states $|00\rangle$ or $|11\rangle$ that remain unchanged. In the middle we have the SWAP which is encoded in the two equations we have found.

The unitary operator we found, is easy to see that it consists of the $\sqrt{\text{iSWAP}}$ in the case in which $\Omega t = \frac{\pi}{4}$ → $t = \frac{\pi}{4\Omega} = \frac{\pi\hbar}{4K}$

$$\hat{U}\Big|_{\Omega t = \frac{\pi}{4}} = \sqrt{\text{iSWAP}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

DI VINCENZO CRITERIA → Criteria for a quantum computer to be operated and be scalable.

Lecture 10 - 02/11/23

1) For any quantum computer we need a SCALABLE and PHYSICAL SYSTEM with WELL DEFINED Q-BIT
2) We have to be able to INITIALITE the Q-BITS
3) READ THE Q-BIT (be able to measure → Stern-Gerlach experiment is an example of READ-operation) In the case of electron spin we'll see the SPIN TO CHARGE CONVERSION.
4) MANIPULATE A UNIVERSAL SET OF QUANTUM GATES the most famous group / set of quantum gates is the CLIFFORD GROUP = [H, S, CNOT] The Clifford group is the minimum set that allows to create entire algorithm with just 3 gates.

↳ we must be able to increase the number of Q-BITS.

↳ must be able to define an initial state and come back to it in case I need

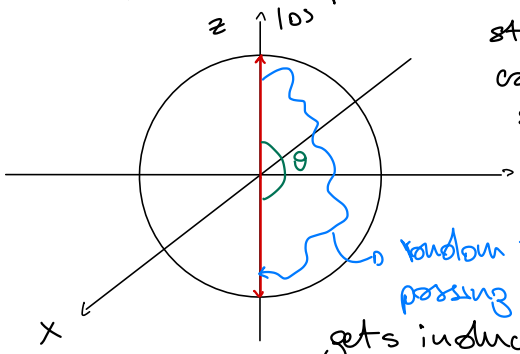
- ATOMIC NUCLEUS
- ELECTRON (energy of the electron is used to establish the state)
- PHOTON (polarization establishes the state)
- SUPERCONDUCTING RESONATOR (harmonic oscillator where the state is defined from a current sign or a frequency)

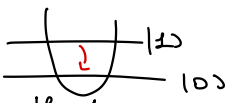
Very controllable and actually one of the most employed q-bit technology.

Rotation by $\frac{\pi}{2}$

All these aspects can be related also to logic computing, BUT the fifth criteria is what makes the difference:

5) LONG DECOHERENCE TIME → decoherence means that at time = 0, when we initialite and then start to manipulate in order to implement the algorithm, we need to operate in a controllable way and, for this reason, the phase must be reliable for the right amount of time in order to properly manipulate the qubits → during operation the qubit phase tends to change → we actuate "PHASING" operations in order to make the phase coherent with the manipulation steps → DECOHERENCE TIME is therefore the period before the qubit phase doesn't result reliable anymore. LONG is referred to the single gate time. I want to maximize the number of gates in the period of time before any qubit looses phase. $N_g = \frac{t_{DECOHERENCE}}{t_g} \approx 1000, 10^9$ (wanted value) There are different components that build up the decoherence time. We have longitudinal and trasversal components. The longitudinal one can be explained by assuming an state starting from $|0\rangle$ and, through a RABI oscillation we can do a rotation around x axis. We can be in a situation like this one with a qubit where due to scattering and the tendency to lose energy, we can have that a qubit state gets altered by a longitudinal rotation like the one described that gets induced by such random effects. The time before such loss of energy $|1\rangle$ is called RELAXATION TIME (T_1).





We then have TRASVERSAL RELAXATION TIME (T_2) that consists in the time needed to loose the phase and perform a change of phase φ along the z axis (in general this is performed by applying a Larmor precession.

NUCLEAR SPIN

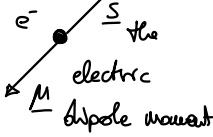
It is not one of the most employed and important spins. But it is one of the most important for biomedical reasons and material characterization.

WHAT IS A NUCLEAR SPIN? We know pretty much everything about the electron. The nucleus is pretty similar, we have protons and neutrons → both of them have a SPIN (they are Fermions)

	Q	SPIN NUMBER
PROTON	+e	$+\frac{1}{2}$
NEUTRON	0	$+\frac{1}{2}$

→ by analyzing the H case we see that:

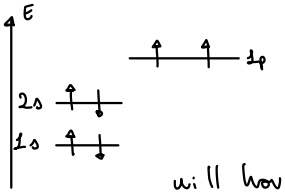
	P	N
^1H	1	0
^2H	1	1
^3H	1	2



S the electric dipole moment is opposed to the spin.

	P	N
^{12}C	6	6
^{16}C	8	8

Generally we have more neutrons than protons since neutrons tend to stabilize the nucleside. For each proton we have an electron with its own spin. For example the ^{12}C we have 6 protons and 6 neutrons. Then we have 6 electrons that occupy the different energy levels:

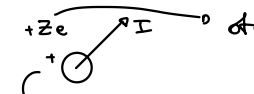


But also in the nucleus we will have energy levels.
As we see, having the SPIN equal to $1/2$, it means to have the possibility of SPIN UP and SPIN DOWN with energies $+\hbar/2$ or $-\hbar/2$. So both protons and neutrons will have a projection of the SPIN along z axis $S_z = \pm\frac{\hbar}{2}$. For instance, having ^{12}C what's the configuration? We define I as the intrinsic SPIN of the nucleons which appears to be null in case of equal number of neutrons and protons (since their spins will compensate for each other). We introduce also ^{28}Si which is the main element for building quantum circuits and it presents an INTRINSIC SPIN equal to 0 → this is very good because the electron spin won't be influenced by nuclear SPIN.

	P	N	I
^1H	1	0	1/2
^2H	1	1	1
^3H	1	2	1/2
^{12}C	6	6	0
^{16}C	8	8	0
^{28}Si	14	14	0
^7Li	3	4	3/2

NOTE: To have zero it is necessary to have an even number of protons equal to an even number of neutrons.

We now assume to have our nucleons instead of the electron.



atomic number → tells we how many protons in the nucleus → therefore the positive charge I have.

positively charged $S^2 = \hbar^2 I(I+1)$

$S_z = \hbar m \rightarrow m = -I, -I+1, \dots, +I-1, +I$

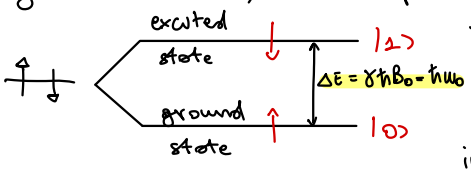
All the rules we applied to electrons can be applied to the nucleus. In this case we talk about NMR → "Nuclear Magnetic Resonance". The nice thing of the nucleides is that they are all different. Electrons instead are all equal and the magnetic dipole moment is defined as: $\underline{\mu} = \gamma \underline{S}$ → γ in case of electrons is defined on the basis of the Bohr magneton $\mu_z = \gamma S_z = \gamma\frac{\hbar}{2} = \mu_B$ In the case of nucleides they present different γ!

For instance $[\gamma] = [10^7 \frac{rad}{T\cdot s}]$ (from Larmor precession frequency $\omega_0 = \gamma B_z$ → $\frac{rad}{s}$, → T)

	P	N	I	γ
^1H	1	0	1/2	26,8
^2H	1	1	1	65
^3H	1	2	1/2	454
^{12}C	6	6	0	
^{16}C	8	8	0	
^{28}Si	14	14	0	
^7Li	3	4	3/2	

When I apply a magnetic field, the different nucleides will do a precession at a different frequency depending on the γ. I can select which nucleide I want to rotate by tuning to the proper frequency.

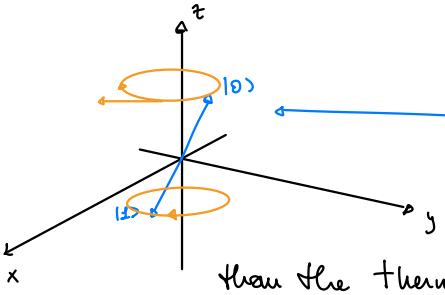
I now take into account the ^1H nucleide where I have the following situation: ↑ ↓ they are degenerate and there's no difference in their energy. If I apply a magnetic field $B_0\cdot\hat{z}$ along the z axis, I can split the two spin energy (Zeeman splitting)



I obtain my basis state but we have a different situation WRT electrons where we used to have $|0\rangle$ on top and $|1\rangle$ on the bottom → this was due to the fact that in the electrons magnetic dipole moment and spin are antiparallel. In the case of nucleons where we have positive charges, the SPIN and magnetic dipole has the same direction as the SPIN. Together with Zeeman splitting I'm also triggering PRECESSION. We remind that the torque $\underline{N}$ is given by $\underline{N} = \underline{\mu} \times \underline{B}$ → this gives us a precession

$\underline{B}$ is along $\hat{z}$ → has the same direction of the state

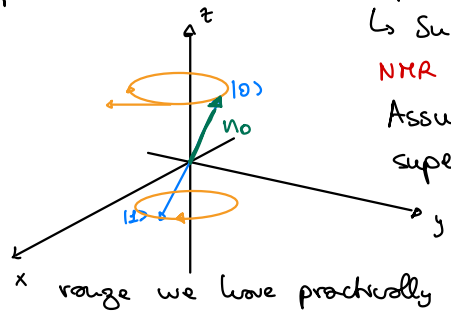
in the clockwise direction (in electrons we had anticlockwise)



We have to consider 2 frames: QUANTUM COMPUTING → we operate at low temperature, therefore the energy splitting will be larger than the thermal excitation kT → $T \ll \Delta E$ → in this case I'm able to use such system as a quantum bit. Now we assume the frame in which we are at ROOM TEMPERATURE → we have $k \simeq \Delta E$ (same range). NOTE: The MRI (Magnetic Resonance Imaging) is performed at ROOM TEMP. In such case we have a sort of dynamic equilibrium that is populating both excited and ground state. Such equilibrium is ruled by BOLTZMANN DISTRIBUTION → $\frac{P_{|1\rangle}}{P_{|0\rangle}} = e^{-\frac{\Delta E}{kT}}$ → decreasing probability to stay in the excited state.

Considering the situation $T \approx \frac{\Delta E}{K} \rightarrow \frac{P_{|1\rangle}}{P_{|0\rangle}} \cong \frac{1}{e^2}$ → The probability to stay in the excited state is $\simeq 1\%$.

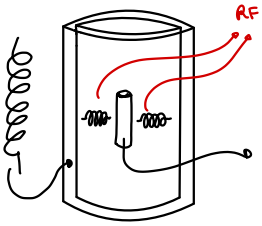
Let's say that the states population is $P_{|1\rangle} \cong 10\%$ and $P_{|0\rangle} \cong 90\%$ → we have an imbalance where we've much more SPIN UP than SPIN DOWN (at thermodynamic equilibrium). From my sample it will appear to have a macroscopic magnetic field pointing up (n_0).



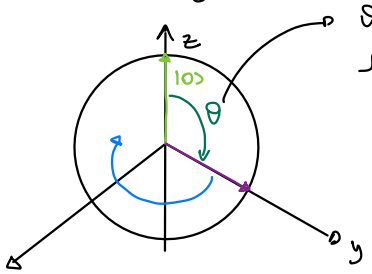
↳ Such MMF is doing precession along z axis.

NMR - Nuclear Magnetic Resonance

Assuming to have a chamber and on the wall of it we have a superconducting solenoid, running along the walls. Such solenoids are kept at liquid He (Helium) temperature (below 2K - 1K). We go under the critical critical point for superconductive behaviour → at such working range we have practically zero resistance → NO DISSIPATION (IDEALLY). We can run as much current as we want without experiencing any voltage. This current into the solenoid generates my $B_0 \hat{z}$ magnetic field (only the DC component along $\hat{z}$)



ω_0 is in the range of RF. With these ESR we can manipulate the Q-BIT. By looking at the block sphere we have the following situation



$\theta = \Omega t$

$\Omega = \frac{\gamma}{2} B_1$ → RABI FREQUENCY

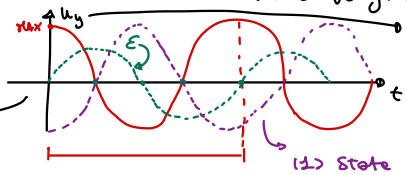
The longer B_1 the faster the rotation

If I want to move of $\theta = \frac{\pi}{2}$ my state, I have to apply for a time $t_{RF} = \frac{\frac{\pi}{2}}{\Omega}$ the electric field.

Once I get my rotation of $\frac{\pi}{2}$ done I will have my state performing a precession along z → and so it will periodically be directed towards $\hat{x}, -\hat{y}, -\hat{x}, \hat{y}$. What I can do now is using the same RF coils to obtain a nuclear SPIN readout.

READ OUT

FARADAY'S INDUCTION LAW: if I have a coil and time varying a magnetic flux, an electro magnetic force is established → $\mathcal{E} = -\frac{d\Phi_B}{dt}$

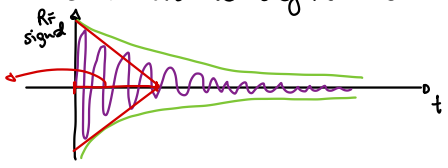


magnetic dipole moment along y. It is MAX at $t = 0 + KT$ (since due to the precession along z we have the dipole periodically oriented along y)

If I do the same rotation for |1⟩ state, I would end up with an anti-phase plot (since the |1⟩ would start from $-\hat{y}$ as MAX point)

↳ I can therefore distinguish |0⟩ and |1⟩ states depending on the behaviour of the electro magnetic force

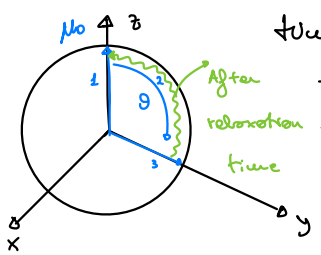
Such measurement is called FREE INDUCTION MEASUREMENT → About this we define the FREE INDUCTION DECAY (FID)

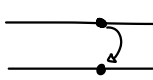


after a certain amount of time (RELAXATION TIME) we end up having $\mathcal{E} = 0$ since the state comes back to the original point.

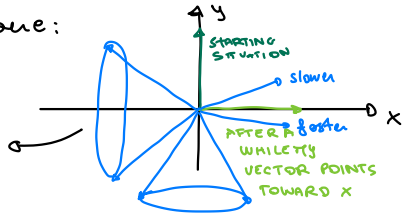
This decay happens because of DECOHERENCE

As we introduced, T_1 = RELAXATION TIME → consists of the time needed to the state to change and come back to the original situation. BUT, in reality the decay doesn't depend from relation time → in fact the decay results to be much faster than that!



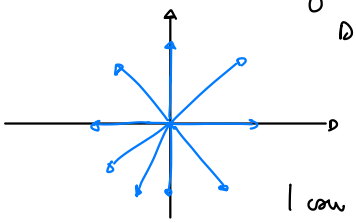


Let's introduce T_2 = TRANSVERSAL DEPHASING TIME and remain in xy plane:



As we keep going on the different components of the macroscopic dipole will be more and more distant since this is a macroscopic dipole moment made of different components, I'll have some interactions that make the electric field NON UNIFORM

This effect gives us back a costellation of dipoles isotropically and randomly distributed that cancel out each other and gives $\varepsilon = 0$. Such dephasing time is the main responsible for FREE INDUCTION DECAY.

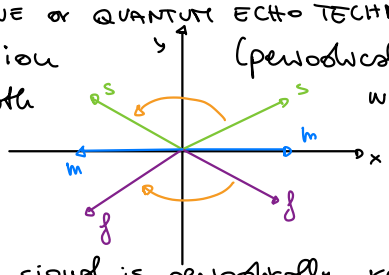


$T_2 << T_1$

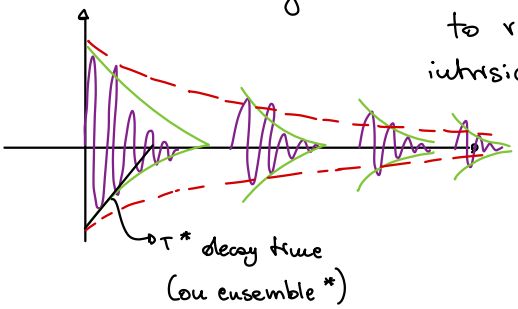
NOTE: T_2 is composed by 2 components, one is related to the distribution of Larmor precession (every nucleous has its own precession frequency). This component can be compensated.

I can rephase → REPHASING TECHNIQUE or QUANTUM ECHO TECHNIQUE:

Such technique consists in performing a rotation (periodically) around y in order to make the fast vector make a longer path with respect to the slower one and therefore compensate the dispersion that would be obtained.



The TDI will therefore be:



we see that thanks to rephasing the TDI signal is periodically represented BUT, as the intrisic decay do to the phase loss is still present, there will be a certain point in time at which the signal is no more present.